AGH University of Krakow

Faculty of Physics and Applied Computer Science

MODULE 2

RELATIVISTIC SYMMETRIES AND FERMIONS

Supporting Lecture Notes for *The Standard Model*

The Guideline

The aim of this module is to understand why the Standard Model must be written as a relativistic theory of fields rather than as a non-relativistic theory of particles. Lorentz symmetry, four-vectors, spinors, gamma matrices, and the Dirac equation are not formal ornaments. They are the minimal ingredients needed to describe spin-$\frac{1}{2}$ matter consistently, to understand why antiparticles appear, and to see how spacetime symmetry constrains the form of particle-physics models.

Prepared for the Standard Model course

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June 8, 2026

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1 Introduction and motivation

The purpose of this module is to develop the minimal relativistic language needed to describe spin-$\frac{1}{2}$ particles consistently and to prepare for the fermionic sector of the Standard Model. At low energies, many quantum systems can be treated successfully within non-relativistic quantum mechanics. Particle physics, however, lies in a regime in which special relativity is not optional. The fields and equations used to describe elementary particles must transform consistently under the symmetry group of spacetime.

This requirement has deep consequences. Once Lorentz symmetry is imposed, the admissible mathematical objects are sharply restricted. Scalars and vectors are not enough to describe all known matter. Electrons, quarks, and neutrinos are spin-$\frac{1}{2}$ particles, and their consistent relativistic description requires the introduction of spinors. From this point follow several central structures of modern particle physics: gamma matrices, the Dirac equation, antiparticles, helicity, chirality, and Lorentz-covariant fermion bilinears.

The logic of the module is therefore structural rather than merely historical. We do not introduce the Dirac equation as an isolated formula to be memorised. Instead, we begin with Minkowski spacetime and Lorentz symmetry, then ask how physical fields transform under that symmetry, why spinors are required, and how the search for a first-order relativistic equation leads to the gamma-matrix algebra and the Dirac equation. Only after this foundation is in place do we discuss free-particle solutions, the interpretation of antiparticles, and the fermionic structures that later enter the Standard Model.

The motivation is twofold. First, relativistic fermion theory is indispensable in its own right: it explains why spin-$\frac{1}{2}$ matter cannot be described adequately by scalar wave equations and why relativistic quantum theory naturally points toward quantum field theory. Second, it prepares the conceptual bridge to gauge theory. The Standard Model is not simply a list of particles and interactions; it is a tightly constrained relativistic quantum field theory built from spinor fields, gauge fields, and symmetry principles. Understanding the fermion sector therefore requires understanding how Lorentz symmetry acts on matter fields before one can meaningfully discuss gauge symmetry, weak interactions, or Yukawa couplings.

The Guideline

Central pedagogical question. What is the minimal relativistic language needed to describe spin-$\frac{1}{2}$ particles consistently and to prepare for the field-theoretic formulation of the Standard Model?
The answer will unfold through the chain

$$\begin{aligned}\text{Minkowski spacetime} &\rightarrow \text{Lorentz symmetry} \rightarrow \text{representations} \rightarrow \text{spinors}\\ &\rightarrow \gamma\text{-matrices} \rightarrow \text{Dirac equation} \rightarrow \text{antiparticles and chirality}\\ &\rightarrow \text{Lorentz-invariant fermion structures.}\end{aligned}$$

1.1 Why relativistic symmetry is unavoidable in particle physics

At low energies, many physical systems can be described successfully by non-relativistic quantum mechanics. In that framework one usually treats the number of particles as fixed and works with a Schrödinger equation that singles out time and treats space differently. This works well for atoms, molecules, and many condensed-matter systems in the appropriate regime. Particle physics is different for at least three reasons.

1. **High energies make relativity unavoidable.** The energies and momenta involved in particle physics are often comparable to or much larger than particle masses, so kinematics must respect Lorentz invariance.

2. **Particle number is not fixed.** Collisions can create and destroy particles. A theory with permanently fixed particle number is too restrictive.

3. **Interactions must be local and covariant.** The natural language is that of fields defined over spacetime, transforming consistently under Lorentz transformations.

The Standard Model is therefore built from relativistic fields. Before discussing gauge symmetry, Higgs fields, or the chiral weak interaction, we must first know what kinds of relativistic objects can exist and how they transform.

1.2 Why Module 2 naturally follows Module 1

Module 1 ended with a bridge to Modules 2 and 3. The point of that bridge was simple but important: once students understand that fields transform in representations and that invariant terms are the admissible building blocks of a Lagrangian, the next natural question is *representations of what*? For internal symmetry this means multiplets under groups such as $\mathrm{SU}(2)$ or $\mathrm{SU}(3)$. For spacetime symmetry it means representations of the Lorentz group. Module 2 is the point where the representation idea becomes concrete for relativistic matter.

A second reason Module 2 follows naturally is that Module 1 already taught the structural importance of invariance. In a relativistic theory, invariance under Lorentz transformations is not optional. It constrains the form of kinetic terms, mass terms, currents, and interactions. Already here we begin to see why not every mathematically imaginable fermion term is allowed in a physical theory.

1.3 Roadmap of these notes

We begin with Minkowski spacetime, four-vectors, invariant intervals, and relativistic kinematics. We then study Lorentz transformations and the structure of their generators. Next we discuss why fields must transform in representations of the Lorentz group and why spin-$\frac{1}{2}$ particles require spinors rather than vectors. That motivates a brief review of the Klein–Gordon equation and the search for a first-order relativistic equation, culminating in the Dirac equation. We then develop the gamma-matrix formalism, solve the free Dirac equation, and discuss spin, helicity, and chirality with special care. After that we turn to antiparticles and the interpretation of relativistic fermion fields. Finally, we discuss Lorentz-covariant bilinears, fermion mass terms, and the bridge from relativistic fermions to the Standard Model.

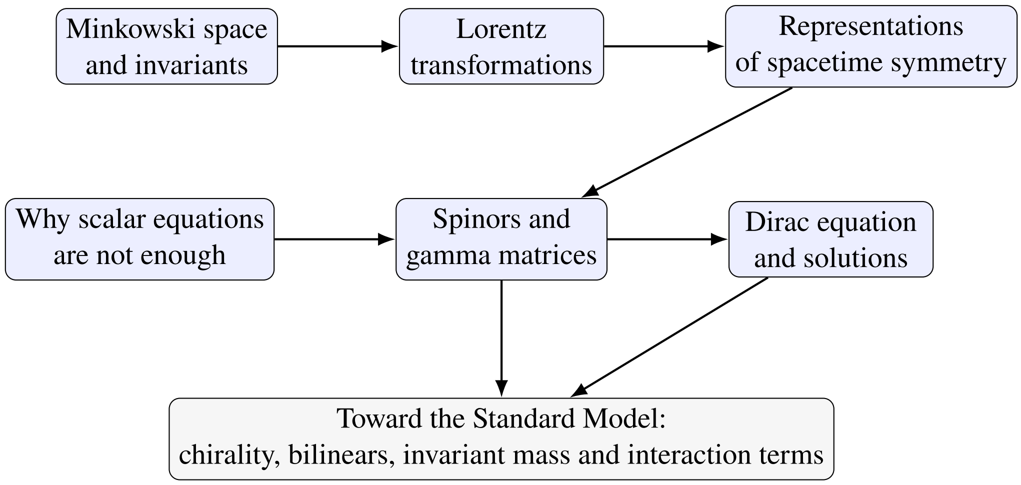

Figure 1: Roadmap of Module 2. The module turns the abstract language of representation theory into a concrete relativistic description of fermion fields.

2 Minkowski spacetime and relativistic kinematics

2.1 Minkowski space as the arena of relativistic physics

The spacetime appropriate to special relativity is four-dimensional Minkowski space. A spacetime point is written as

$$x^\mu = (x^0, x^1, x^2, x^3) = (t, \mathbf{x})$$

in natural units with $\hbar = c = 1$ unless otherwise stated. The index μ runs over $0, 1, 2, 3$. Different sign conventions for the Minkowski metric are used in the literature. In these notes we adopt the mostly-minus convention

$$\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

This convention is standard in many particle-physics texts and is also consistent with the notation used in the attached AGH slide deck when writing $p^\mu p_\mu = m^2$.

Definition 2.1: Minkowski metric and invariant interval

The Minkowski metric $\eta_{\mu\nu}$ defines the invariant spacetime interval

$$\mathrm{d}s^2 = \eta_{\mu\nu}\, \mathrm{d}x^\mu \mathrm{d}x^\nu = \mathrm{d}t^2 - \mathrm{d}\mathbf{x}^2.$$

A transformation that preserves this interval is called a Lorentz transformation.

The key word is *invariant*. Two inertial observers may assign different coordinates to the same event, but they agree on the value of $\mathrm{d}s^2$. This is the spacetime analogue of how Euclidean rotations preserve lengths in ordinary three-dimensional geometry.

2.2 Contravariant and covariant components

The metric allows us to raise and lower indices:

$$x_\mu = \eta_{\mu\nu} x^\nu, \qquad x^\mu = \eta^{\mu\nu} x_\nu.$$

Because the inverse metric is numerically identical to the metric for our choice of signature,

$$\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1),$$

one obtains

$$x_\mu = (t, -x, -y, -z).$$

Thus upper and lower indices are not typographical decoration. They carry information about how components transform and how invariant contractions are formed.

Example 2.1: A basic invariant contraction

For two four-vectors a^μ and b^μ one defines the Lorentz scalar

$$a \cdot b \equiv a^\mu b_\mu = a^0 b^0 - \mathbf{a} \cdot \mathbf{b}.$$

This quantity has the same numerical value in every inertial frame. That is why scalar products are central in relativistic model building.

2.3 Four-vectors and Lorentz scalars

A *four-vector* is an object whose components transform linearly under Lorentz transformations:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu, \qquad \Lambda^\mu{}_\nu = \begin{pmatrix} \Lambda^0{}_0 & \Lambda^0{}_1 & \Lambda^0{}_2 & \Lambda^0{}_3 \\ \Lambda^1{}_0 & \Lambda^1{}_1 & \Lambda^1{}_2 & \Lambda^1{}_3 \\ \Lambda^2{}_0 & \Lambda^2{}_1 & \Lambda^2{}_2 & \Lambda^2{}_3 \\ \Lambda^3{}_0 & \Lambda^3{}_1 & \Lambda^3{}_2 & \Lambda^3{}_3 \end{pmatrix}.$$

Here $\Lambda^\mu{}_\nu$ is the Lorentz transformation matrix, i.e. the linear map that preserves the Minkowski interval. The most important examples are spacetime position x^μ, momentum p^μ, and derivatives ∂_μ. A *Lorentz scalar* is a quantity unchanged by the transformation, such as $x^\mu x_\mu$, $p^\mu p_\mu$, or the action of a scalar field evaluated at the transformed point.

The energy-momentum four-vector is written as

$$p^\mu = (E, \mathbf{p}), \qquad p_\mu = (E, -\mathbf{p}).$$

Its invariant norm is

$$p^\mu p_\mu = E^2 - \mathbf{p}^2.$$

For an on-shell free particle of mass m, this satisfies the mass-shell relation

$$p^\mu p_\mu = m^2, \qquad \text{equivalently} \qquad E^2 = \mathbf{p}^2 + m^2.$$

This equation is not merely kinematics. It is one of the deepest clues that any relativistic quantum equation must incorporate.

2.4 Proper time and timelike, lightlike, spacelike intervals

Depending on the sign of $\mathrm{d}s^2$, intervals fall into three classes:

- $\mathrm{d}s^2 > 0$: timelike,
- $\mathrm{d}s^2 = 0$: lightlike or null,
- $\mathrm{d}s^2 < 0$: spacelike.

For a timelike worldline one defines proper time by

$$\mathrm{d}\tau^2 = \mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}\mathbf{x}^2.$$

Proper time is the time measured in the instantaneous rest frame of the particle. The four-velocity is then

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}, \qquad u^\mu u_\mu = 1,$$

and the four-momentum of a particle of mass m is

$$p^\mu = mu^\mu.$$

This immediately yields the invariant mass relation.

Remark 2.1: Why proper time matters

Proper time is not just a nice invariant parameter. It is the clearest reminder that in relativity the physically meaningful notion of elapsed time is observer dependent unless expressed in invariant form. Later, when we discuss particle worldlines and free-particle phases $\mathrm{e}^{-\mathrm{i}p\cdot x}$, invariant combinations like $p \cdot x$ are what survive across frames.

2.5 Examples of relativistic invariants

The formalism begins to feel more natural once one sees how many physical quantities are built from invariant contractions. Examples include:

$$\begin{aligned}
x^\mu x_\mu &= t^2 - \mathbf{x}^2, \\
p^\mu p_\mu &= m^2, \\
p^\mu x_\mu &= Et - \mathbf{p}\cdot\mathbf{x}, \\
\partial^\mu \partial_\mu &= \Box = \frac{\partial^2}{\partial t^2} - \nabla^2,
\end{aligned}$$

in which

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right), \qquad \partial^\mu \equiv \eta^{\mu\nu}\partial_\nu = \left(\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right).$$

The d'Alembertian operator $\Box$ will appear repeatedly in relativistic wave equations. The phase $p \cdot x$ determines plane-wave solutions. The invariant norm of momentum determines the particle mass. At every step the rule is the same: allowed equations must be built from Lorentz-covariant objects in such a way that the full expression transforms consistently.

Take-home message

Special relativity does not merely modify a few kinematic formulas. It changes the admissible mathematical form of the theory. The central requirement is not that equations "look relativistic", but that they are built from objects that transform consistently under Lorentz transformations and that physically relevant quantities are expressed through invariant combinations.

3 Lorentz transformations

3.1 Definition and metric-preserving condition

A Lorentz transformation is a linear map

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu},$$

where $\Lambda^{\mu}{}_{\nu}$ is the Lorentz transformation matrix. In general,

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \Lambda^0{}_0 & \Lambda^0{}_1 & \Lambda^0{}_2 & \Lambda^0{}_3 \\ \Lambda^1{}_0 & \Lambda^1{}_1 & \Lambda^1{}_2 & \Lambda^1{}_3 \\ \Lambda^2{}_0 & \Lambda^2{}_1 & \Lambda^2{}_2 & \Lambda^2{}_3 \\ \Lambda^3{}_0 & \Lambda^3{}_1 & \Lambda^3{}_2 & \Lambda^3{}_3 \end{pmatrix}.$$

It preserves the Minkowski interval. Imposing

$$x'^{\mu} x'_{\mu} = x^{\mu} x_{\mu}$$

for all vectors x^{μ} gives the defining condition

$$\Lambda^T \eta \, \Lambda = \eta.$$

This is the analogue of $R^{\mathrm{T}} R = \mathbf{1}$ for Euclidean rotations, except that the Minkowski metric replaces the Euclidean identity matrix.

The full Lorentz group has disconnected components. For most purposes in field theory one focuses on the subgroup continuously connected to the identity, known as the proper orthochronous Lorentz group and often denoted $\mathrm{SO}^{+}(1,3)$. This excludes parity and time reversal, which are discrete transformations and will only be mentioned briefly here.

3.2 Rotations and boosts

Lorentz transformations come in two familiar continuous types.

Spatial rotations mix spatial coordinates while leaving time unchanged. For example, a rotation about the z-axis by angle ϕ acts as

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}.$$

Boosts mix time with one spatial direction. A boost along the x-axis with velocity v is

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}, \qquad \gamma = \frac{1}{\sqrt{1 - v^2}}.$$

Unlike rotations, boosts are non-compact: one can keep boosting without coming back to the identity.

Example 3.1: Rapidity

It is often convenient to parameterise a boost by rapidity ξ rather than velocity. Define

$$\gamma = \cosh\xi, \qquad \gamma v = \sinh\xi, \qquad v = \tanh\xi.$$

Then the boost matrix takes the compact form

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\xi & -\sinh\xi \\ -\sinh\xi & \cosh\xi \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}.$$

Rapidity adds linearly under successive collinear boosts, which makes it conceptually cleaner than velocity in many calculations.

3.3 Infinitesimal Lorentz transformations

To study generators and algebra, one looks near the identity. Write

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu, \qquad |\omega^\mu{}_\nu| \ll 1.$$

Substituting into $\Lambda^{\mathrm{T}}\eta\Lambda = \eta$ and keeping only first order terms yields

$$\omega_{\mu\nu} + \omega_{\nu\mu} = 0,$$

where

$$\omega_{\mu\nu} = \eta_{\mu\rho}\, \omega^\rho{}_\nu.$$

Thus the infinitesimal parameters form an antisymmetric tensor. In four dimensions an antisymmetric 4×4 matrix has six independent components, corresponding to three rotations and three boosts.

A general infinitesimal Lorentz transformation can therefore be written as

$$\Lambda = \mathbf{1} - \frac{\mathrm{i}}{2}\omega_{\mu\nu} M^{\mu\nu},$$

where $M^{\mu\nu} = -M^{\nu\mu}$ are the generators in the appropriate representation.

3.4 Lorentz generators and Lorentz algebra

In the vector representation the generators satisfy the Lorentz algebra

$$[M^{\mu\nu}, M^{\rho\sigma}] = \mathrm{i}\,(\eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\mu\rho} + \eta^{\mu\sigma}M^{\nu\rho})\,. \tag{1}$$

This compact equation contains the full structure of spacetime symmetry in special relativity.

It is common to separate the generators into rotations and boosts:

$$J^i = \frac{1}{2}\epsilon^{ijk}M^{jk}, \qquad K^i = M^{0i}.$$

Their commutators are

$$\left[J^i, J^j\right] = \mathrm{i}\epsilon^{ijk}J^k, \tag{2}$$
$$\left[J^i, K^j\right] = \mathrm{i}\epsilon^{ijk}K^k, \tag{3}$$
$$\left[K^i, K^j\right] = -\mathrm{i}\epsilon^{ijk}J^k. \tag{4}$$

The minus sign in the boost-boost commutator is the algebraic trace of the non-compact nature of Lorentz symmetry.

Remark 3.1: What the generators mean physically

Generators are the infinitesimal version of transformations. Rotations tell us how objects respond to changes of spatial orientation. Boosts tell us how they respond to changes of inertial frame. The representation carried by a field determines *how* these generators act. That is exactly why representation theory becomes physics in this context.

3.5 A brief comment on the Poincaré group

The full continuous symmetry group of Minkowski space is the Poincaré group: Lorentz transformations together with spacetime translations. Its generators are $M^{\mu\nu}$ and P^μ, with

$$[P^\mu, P^\nu] = 0, \qquad [M^{\mu\nu}, P^\rho] = \mathrm{i}\,(\eta^{\nu\rho}P^\mu - \eta^{\mu\rho}P^\nu)\,.$$

For the purposes of the present module, the key point is simple: fields and particle states must transform consistently under the symmetry of spacetime. The Lorentz part will be our main focus, while the Poincaré perspective becomes useful when classifying particles by mass and spin.

3.6 A short remark on particle classification

At the level of physical states, particles are classified by unitary irreducible representations of the Poincaré group. For massive particles one may go to the rest frame $p^\mu = (m, 0, 0, 0)$, and the little group is then the rotation group. This is why massive particles carry ordinary spin labels $j = 0, \frac{1}{2}, 1, \ldots$. For massless particles there is no rest frame, and the relevant label is helicity. We will not develop Wigner's classification in detail here, but the message is worth keeping in mind: the spin carried by a particle is not an ad hoc label. It emerges from symmetry.

4 Representations of Lorentz symmetry

4.1 Why fields must transform in representations

Module 1 already introduced the essential idea that a group becomes physically meaningful only when it acts on something. Here the relevant group is the Lorentz group, and the "somethings" are fields. A scalar field transforms trivially, a vector field transforms like a four-vector, and higher-rank tensors transform with one Lorentz matrix for each index. So far, this looks familiar. But matter fields in the Standard Model are not scalars or vectors. Electrons, quarks, and neutrinos are spin-$\frac{1}{2}$ objects. To describe them relativistically, we need a new kind of representation.

Why is this not optional? Because the transformation law of a field determines whether an equation involving that field is Lorentz covariant. If the field transforms incorrectly, then even a beautifully written equation can fail to describe the same physics in different inertial frames.

Definition 4.1: Lorentz covariance of a field equation

A field equation is Lorentz covariant if, whenever a field configuration solves the equation in one inertial frame, the transformed field solves the transformed equation in any other inertial frame. Covariance is a statement about the transformation of both the coordinates and the field itself.

4.2 Scalars, vectors, tensors

A scalar field $\phi(x)$ transforms as

$$\phi'(x') = \phi(x).$$

A vector field $A^\mu(x)$ transforms as

$$A'^\mu(x') = \Lambda^\mu{}_\nu A^\nu(x).$$

A rank-two tensor $T^{\mu\nu}(x)$ transforms with two Lorentz matrices:

$$T'^{\mu\nu}(x') = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma T^{\rho\sigma}(x).$$

These are all honest tensor representations of the Lorentz group. But none of them describes an elementary spin-$\frac{1}{2}$ matter field.

4.3 Why spin-half needs more than vectors

A vector changes sign only under a rotation by angle π when the geometry dictates it. A spinor is different: under a full 2π rotation it picks up a minus sign, and only after a 4π rotation does it return to itself. This is not a curiosity of notation. It is the signature of a fundamentally different representation of spacetime symmetry.

More formally, the proper Lorentz group $\mathrm{SO}^+(1,3)$ does not itself admit ordinary single-valued spinor representations; spinors belong to representations of its double cover, $\mathrm{Spin}(1,3)$. In four dimensions one has the local isomorphism

$$\mathrm{Spin}(1,3) \cong \mathrm{SL}(2,\mathbb{C}).$$

This is the representation-theoretic origin of two-component Weyl spinors and, later, four-component

Dirac spinors.

Remark 4.1: How much group theory do we really need?

For the purposes of this module, students do *not* need a fully abstract treatment of $\mathrm{SL}(2,\mathbb{C})$. What matters is the logic: the ordinary four-vector representation is not the most basic one for relativistic matter. The fundamental building blocks are two inequivalent two-component spinor representations, often called left-handed and right-handed Weyl spinors. Dirac spinors are built from them.

4.4 The two Weyl representations in words

The Lorentz algebra can be reorganised in terms of two commuting copies of an $\mathfrak{su}(2)$-like structure. This leads to representations labelled by pairs (j_L, j_R). Then

$$(0,0) \quad \text{is a scalar,} \qquad \left(\tfrac{1}{2},\tfrac{1}{2}\right) \quad \text{is a four-vector,}$$

while the two basic spinor representations are

$$\left(\tfrac{1}{2},0\right) \qquad \text{and} \qquad \left(0,\tfrac{1}{2}\right).$$

These are the left-handed and right-handed Weyl spinors. A Dirac spinor is their direct sum,

$$\left(\tfrac{1}{2},0\right) \oplus \left(0,\tfrac{1}{2}\right).$$

One should not over-interpret the labels at first encounter. The main point is that spinors are not vectors in disguise. They are distinct spacetime representations.

Take-home message

When one says that a field "has spin", one is really saying that it belongs to a particular representation of spacetime symmetry. Scalars, vectors, and spinors are not just different types of indices. They are different transformation laws, and therefore different kinds of physical objects.

5 Why spinors are required

5.1 The historical problem of relativistic wave equations

Suppose we try to describe a free relativistic particle quantum mechanically. The relativistic energy-momentum relation is

$$E^2 = \mathbf{p}^2 + m^2.$$

In canonical quantisation one makes the substitutions

$$E \to \mathrm{i}\frac{\partial}{\partial t}, \qquad \mathbf{p} \to -\mathrm{i}\nabla.$$

Substituting these into the relativistic relation gives

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\phi = 0,$$

or, more compactly,

$$(\Box + m^2)\phi = 0. \tag{5}$$

This is the Klein–Gordon equation. It is certainly relativistic, because it is built from Lorentz scalars. But it does not solve the problem of relativistic spin-$\frac{1}{2}$ matter.

5.2 Why the Klein–Gordon equation is not enough for fermions

The Klein–Gordon equation describes spin-0 degrees of freedom very naturally. It is second order in time derivatives, and its probability interpretation is not as straightforward as in non-relativistic quantum mechanics. More importantly for us, it does not encode intrinsic spin-$\frac{1}{2}$. One can force each component of a multi-component object to satisfy a Klein–Gordon equation, but that still does not explain how the components must transform under the Lorentz group or why spin-$\frac{1}{2}$ should appear.

What we want is an equation with the following properties:

- it should be Lorentz covariant,
- it should be first order in time derivatives, more in the spirit of the Schrödinger equation,
- it should reproduce the relativistic dispersion relation,
- it should describe spin-$\frac{1}{2}$ matter,
- it should lead naturally to a conserved current.

The Dirac equation is the answer.

5.3 Factorisation logic

Dirac's key idea was to seek a first-order operator whose square reproduces the Klein–Gordon operator. One starts from the Hamiltonian form

$$\mathrm{i}\partial_t\psi = H\psi, \qquad H = \boldsymbol{\alpha}\cdot(-\mathrm{i}\nabla) + \beta m,$$

with unknown coefficients α^i and β. The requirement is that every component of ψ should then satisfy the relativistic dispersion relation

$$E^2 = \mathbf{p}^2 + m^2.$$

Squaring the Hamiltonian gives

$$H^2 = \alpha^i\alpha^j p_i p_j + m\big(\alpha^i\beta + \beta\alpha^i\big)p_i + \beta^2 m^2.$$

Because $p_i p_j$ is symmetric in i, j, only the anticommutator part of $\alpha^i\alpha^j$ contributes. Matching H^2 to $\mathbf{p}^2 + m^2$ therefore requires

$$\big\{\alpha^i, \alpha^j\big\} = 2\delta^{ij}, \qquad \big\{\alpha^i, \beta\big\} = 0, \qquad \beta^2 = \mathbf{1}. \tag{6}$$

These relations cannot be satisfied by ordinary numbers. They force α^i and β to be matrices acting on a multi-component wavefunction. In other words, relativity and linearity already demand internal spinor structure.

Introducing

$$\gamma^0 \equiv \beta, \qquad \gamma^i \equiv \beta\alpha^i,$$

one may rewrite (6) as the covariant Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbf{1}.$$

The upper-index object $\eta^{\mu\nu}$ is the inverse Minkowski metric, defined by

$$\eta^{\mu\rho}\eta_{\rho\nu} = \delta^\mu{}_\nu.$$

With the mostly-minus convention used here,

$$\eta^{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1).$$

Thus the factorisation problem is not a side remark. It is the route by which the gamma-matrix algebra and the Dirac spinor are forced upon us.

Example 5.1: Why matrices are forced on us

Suppose we try to write

$$(\mathrm{i}\partial_t - A^i(-\mathrm{i}\partial_i) - Bm)\psi = 0$$

for some coefficients A^i and B. Squaring the operator should reproduce $(\Box + m^2)\psi = 0$. This is only possible if the coefficients satisfy nontrivial algebraic relations such as

$$\left\{A^i, A^j\right\} = 2\delta^{ij}, \qquad \left\{A^i, B\right\} = 0, \qquad B^2 = 1.$$

Ordinary numbers cannot do this. Matrices can. This is the algebraic heart of the Dirac construction.

6 Weyl spinors, Dirac spinors, and basic spinor structure

6.1 Two-component spinors as the basic relativistic objects

A two-component Weyl spinor is the simplest nontrivial spinor representation in four-dimensional relativistic theory. One usually distinguishes a left-handed spinor ψ_L and a right-handed spinor ψ_R. The two are inequivalent as Lorentz representations, although they are related by complex conjugation in an appropriate sense.

At this stage it is helpful to think operationally. A spinor is an object that transforms linearly under Lorentz transformations, but not as a scalar or vector. The matrices acting on it are 2×2 rather than 4×4 for the Weyl case. In many modern Standard Model discussions, especially of weak interactions, the two-component language is conceptually the cleanest one.

6.2 Left-handed and right-handed Weyl spinors

The names “left-handed” and “right-handed” here refer first to Lorentz representation type, not yet to helicity of a moving particle. That distinction matters and will be discussed carefully later. Very roughly:

- a left-handed Weyl spinor transforms in the $\left(\frac{1}{2}, 0\right)$ representation,
- a right-handed Weyl spinor transforms in the $\left(0, \frac{1}{2}\right)$ representation.

A massless relativistic theory may naturally be written in terms of independent Weyl spinors. A massive Dirac fermion requires both chiralities if one wants a Lorentz-invariant mass term.

6.3 Building a Dirac spinor

A four-component Dirac spinor is constructed by combining a left-handed and a right-handed Weyl spinor into a single object,

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}.$$

This is often called the chiral or Weyl basis. The advantage of this basis is that the two chiral components are displayed explicitly.

One reason Dirac spinors are pedagogically convenient is that they allow the entire free relativistic fermion theory to be written compactly in terms of gamma matrices. But it is important not to lose sight of the more fundamental statement: a Dirac spinor is not an irreducible Lorentz representation. It is a reducible object built from two irreducible Weyl pieces.

Remark 6.1: Why the distinction matters already here

In non-relativistic quantum mechanics one often speaks loosely of “the spinor” of an electron. In relativistic theory that language is too imprecise. One should know whether one means a Weyl spinor, a Dirac spinor, or later perhaps a Majorana spinor. The distinctions are not cosmetic; they affect transformation properties, mass terms, and interactions.

6.4 A brief forward-looking remark on Majorana spinors

In four-dimensional relativistic theory one may also impose a reality condition relating a spinor to its charge conjugate. This leads to the idea of a Majorana spinor. Such spinors are important in some beyond-the-Standard-Model contexts and in discussions of neutrino masses. However, because the minimal Standard Model is chiral and because the pedagogical focus of this module is on Lorentz symmetry, Weyl and Dirac spinors will be our main tools. We will mention Majorana structure only when it clarifies the logic of possible fermion mass terms.

7 Gamma matrices and Clifford algebra

7.1 Definition of gamma matrices

The Dirac equation is built from four matrices γ^μ satisfying the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbf{1}. \tag{7}$$

These are the gamma matrices. They are representation dependent in their explicit matrix form, but their algebraic relations are representation independent.

In the chiral basis one convenient choice is

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix},$$

where

$$\sigma^0 = \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma^\mu = (\sigma^0, \sigma^i), \qquad \bar{\sigma}^\mu = (\sigma^0, -\sigma^i),$$

and σ^i are the Pauli matrices,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Equivalently, the gamma matrices in the chiral basis are

$$\gamma^0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

7.2 Why the Clifford algebra matters

The point of the Clifford algebra is that it makes possible a first-order relativistic spinor equation whose square reproduces the Klein–Gordon operator. Anticipating the equation that will shortly be identified as the Dirac equation, consider the first-order operator

$$\mathrm{i}\gamma^\mu\partial_\mu - m.$$

Its product with the conjugate-sign operator is

$$(\mathrm{i}\gamma^\mu\partial_\mu - m)(\mathrm{i}\gamma^\nu\partial_\nu + m) = -\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu - m^2,$$

and because partial derivatives commute, only the symmetric anticommutator part contributes:

$$-\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\,\partial_\mu\partial_\nu - m^2 = -(\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2) = -(\Box + m^2).$$

Thus, if a spinor field satisfies the first-order equation

$$(\mathrm{i}\gamma^\mu \partial_\mu - m)\psi = 0,$$

then each component automatically satisfies the Klein–Gordon equation. This is why the Clifford algebra is exactly the algebraic structure needed for the relativistic spinor equation that we will formally introduce in the next section as the Dirac equation.

The converse is not true: the Klein–Gordon equation by itself does not encode the full spinor structure.

7.3 Dirac adjoint

The ordinary Hermitian conjugate $\psi^\dagger$ is not itself the right object for forming Lorentz scalars from Dirac spinors. One instead defines the Dirac adjoint

$$\bar{\psi} \equiv \psi^\dagger \gamma^0.$$

With this definition, bilinears such as $\bar{\psi}\psi$ and $\bar{\psi}\gamma^\mu\psi$ transform as Lorentz scalar and vector respectively.

Definition 7.1: Dirac adjoint

For a Dirac spinor ψ, the Dirac adjoint is

$$\bar{\psi} = \psi^\dagger \gamma^0.$$

This is the natural object with which to build Lorentz-covariant fermion bilinears.

7.4 The chiral matrix gamma-five

A central object in relativistic fermion theory is

$$\gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3.$$

In the chiral basis this takes the simple form

$$\gamma^5 = \begin{pmatrix} -\mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}.$$

It obeys

$$\{\gamma^5, \gamma^\mu\} = 0, \qquad (\gamma^5)^2 = \mathbf{1}.$$

Its eigenvalues are ± 1, which allows us to define chiral projection operators,

$$P_L = \frac{1}{2}(1 - \gamma^5), \qquad P_R = \frac{1}{2}(1 + \gamma^5). \tag{8}$$

These satisfy the usual projector relations

$$P_L^2 = P_L, \qquad P_R^2 = P_R, \qquad P_L P_R = P_R P_L = 0, \qquad P_L + P_R = \mathbf{1}.$$

Because γ^5 anticommutes with γ^μ, gamma matrices flip chirality:

$$\gamma^\mu P_L = P_R \gamma^\mu, \qquad \gamma^\mu P_R = P_L \gamma^\mu. \tag{9}$$

This simple identity is used repeatedly later. It explains, for example, why the kinetic term connects the derivative to the same chiral sector while the mass term couples opposite chiralities.

Then

$$\psi_L = P_L \psi, \qquad \psi_R = P_R \psi, \qquad \psi = \psi_L + \psi_R.$$

These are the left- and right-chiral components of a Dirac spinor.

7.5 Lorentz generators in spinor space

The generators of Lorentz transformations acting on Dirac spinors are

$$\Sigma^{\mu\nu} = \frac{\mathrm{i}}{4} \left[\gamma^\mu, \gamma^\nu\right]. \tag{10}$$

For an infinitesimal Lorentz transformation one has

$$\psi \to \left(1 - \frac{\mathrm{i}}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\right)\psi.$$

This is the spinor analogue of how vectors transform with matrices generated by $M^{\mu\nu}$. The explicit matrices differ, but the underlying algebra is the same.

Remark 7.1: Convention dependence and physics

The explicit appearance of gamma matrices depends on the chosen representation – Dirac basis, Weyl basis, Majorana basis, and so on. Physical statements do not. Whether one writes γ^5 as diagonal or off-diagonal is a matter of representation. Whether chirality can be projected, whether bilinears transform as scalar or vector, and whether the Dirac equation is Lorentz covariant are physical statements and therefore basis independent.

8 Derivation and structure of the Dirac equation

8.1 From the relativistic dispersion relation to the Dirac equation

The cleanest derivation begins from the Hamiltonian ansatz already motivated above,

$$\mathrm{i}\partial_t \psi = \left(-\mathrm{i}\alpha^i \partial_i + \beta m\right)\psi, \tag{11}$$

with matrix coefficients satisfying (6). Writing this as

$$\left(\mathrm{i}\partial_t + \mathrm{i}\alpha^i \partial_i - \beta m\right)\psi = 0$$

and multiplying by β from the left gives

$$\left(\mathrm{i}\beta\partial_t + \mathrm{i}\beta\alpha^i \partial_i - m\right)\psi = 0.$$

With the definitions

$$\gamma^0 \equiv \beta, \qquad \gamma^i \equiv \beta\alpha^i,$$

this becomes

$$(\mathrm{i}\gamma^\mu \partial_\mu - m)\psi = 0. \tag{12}$$

This is the free Dirac equation.

We adopt this Hamiltonian ansatz (Eq.11) because the goal is to construct a relativistic equation that is first order in time, Schrödinger-like in form, and whose square reproduces the relativistic mass-shell relation. This naturally suggests a Hamiltonian linear in momentum.

The conceptual gain is substantial. Equation (12) is first order in time derivatives, is compatible with the relativistic dispersion relation, and is written entirely in Lorentz-covariant language. The price is that ψ must be a multi-component spinor field and the coefficients γ^μ must satisfy the Clifford algebra.

To see explicitly that the Dirac equation reproduces the relativistic mass-shell condition, apply the conjugate operator from the left:

$$(\mathrm{i}\gamma^\mu \partial_\mu + m)(\mathrm{i}\gamma^\nu \partial_\nu - m)\psi = 0.$$

Using the commutativity of partial derivatives and the anticommutation relations of the gamma matrices, one finds

$$(\Box + m^2)\psi = 0.$$

Thus every component of a free Dirac spinor satisfies the Klein–Gordon equation. The Dirac equation is therefore a linear, spinor-valued square root of the relativistic dispersion relation, not an unrelated equation written by guesswork.

8.2 Hamiltonian form

Separating time and space, the Dirac equation can be written as

$$\mathrm{i}\frac{\partial\psi}{\partial t} = \left(-\mathrm{i}\alpha^i \partial_i + \beta m\right)\psi,$$

where

$$\alpha^i = \gamma^0\gamma^i, \qquad \beta = \gamma^0.$$

This makes it look more like a relativistic version of the Schrödinger equation, with a Hamiltonian linear in momentum.

8.3 Adjoint equation

Starting from the Dirac equation and taking the Hermitian conjugate carefully, one finds

$$-\mathrm{i}(\partial_\mu \psi^\dagger)(\gamma^\mu)^\dagger - m\psi^\dagger = 0.$$

Now use the standard identity

$$\gamma^0(\gamma^\mu)^\dagger\gamma^0 = \gamma^\mu,$$

multiply from the right by γ^0, and introduce $\bar{\psi} = \psi^\dagger \gamma^0$. The result is the adjoint equation

$$\bar{\psi}(\mathrm{i}\overleftarrow{\partial}_\mu \gamma^\mu + m) = 0, \tag{13}$$

where the left arrow reminds us that the derivative acts on $\bar{\psi}$ rather than on anything to its right. This equation is not an optional companion formula; it is the ingredient needed to derive conserved bilinears and to write the fermion action in a manifestly covariant way.

8.4 Probability current and continuity equation

A major advantage of the Dirac equation over the Klein–Gordon equation is that it leads naturally to a conserved current whose time component is non-negative at the wavefunction level, $j^0 = \psi^\dagger \psi$. Multiply the Dirac equation on the left by $\bar{\psi}$, and the adjoint equation on the right by ψ:

$$\bar{\psi}\,\mathrm{i}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi = 0,$$

$$\mathrm{i}(\partial_\mu \bar{\psi})\gamma^\mu \psi + m\bar{\psi}\psi = 0.$$

Adding them gives

$$\mathrm{i}(\partial_\mu \bar{\psi})\gamma^\mu \psi + \bar{\psi}\,\mathrm{i}\gamma^\mu \partial_\mu \psi = 0.$$

Since the gamma matrices are constant, this is

$$\mathrm{i}\,\partial_\mu(\bar{\psi}\gamma^\mu \psi) = 0,$$

and therefore

$$\partial_\mu(\bar{\psi}\gamma^\mu \psi) = 0.$$

Thus the Dirac current is

$$j^\mu = \bar{\psi}\gamma^\mu \psi, \qquad \partial_\mu j^\mu = 0. \tag{14}$$

Its time component is

$$j^0 = \psi^\dagger \psi,$$

which is non-negative. This is one reason the Dirac equation provides a much more satisfactory single-particle probability interpretation than the Klein–Gordon equation.

Take-home message

The Dirac equation does three jobs at once: it is Lorentz covariant, it is first order in derivatives, and it leads to a conserved current with a natural positive density. None of these properties is accidental. They are the direct consequence of asking for a consistent relativistic description of spin-$\frac{1}{2}$ matter.

9 Solutions of the Dirac equation

9.1 Plane-wave ansatz

For a free particle it is natural to look for plane-wave solutions. We therefore try spinor fields of the form

$$\psi(x) = u(p)\,\mathrm{e}^{-\mathrm{i}p\cdot x} \qquad \text{or} \qquad \psi(x) = v(p)\,\mathrm{e}^{+\mathrm{i}p\cdot x},$$

where

$$p \cdot x = p_\mu x^\mu = Et - \mathbf{p} \cdot \mathbf{x},$$

and where $u(p)$ and $v(p)$ are constant spinors, independent of spacetime position.

It is convenient to introduce the slash notation

$$\not{p} \equiv \gamma^\mu p_\mu.$$

Substituting the first ansatz into the free Dirac equation

$$(\mathrm{i}\gamma^\mu \partial_\mu - m)\psi(x) = 0$$

gives

$$(\mathrm{i}\gamma^\mu \partial_\mu - m)\big(u(p)\,\mathrm{e}^{-\mathrm{i}p\cdot x}\big) = 0.$$

Since $u(p)$ is constant, the derivative acts only on the exponential:

$$\partial_\mu \mathrm{e}^{-\mathrm{i}p\cdot x} = -\mathrm{i}p_\mu \mathrm{e}^{-\mathrm{i}p\cdot x}.$$

Therefore

$$(\mathrm{i}\gamma^\mu \partial_\mu - m)\big(u(p)\,\mathrm{e}^{-\mathrm{i}p\cdot x}\big) = \big(\gamma^\mu p_\mu - m\big)u(p)\,\mathrm{e}^{-\mathrm{i}p\cdot x}.$$

Because the exponential is never zero, this implies

$$(\gamma^\mu p_\mu - m)u(p) = 0,$$

or, in slash notation,

$$(\not{p} - m)u(p) = 0.$$

For the second ansatz,

$$\psi(x) = v(p)\,\mathrm{e}^{+\mathrm{i}p\cdot x},$$

one similarly finds

$$\partial_\mu \mathrm{e}^{+\mathrm{i}p\cdot x} = +\mathrm{i}p_\mu \mathrm{e}^{+\mathrm{i}p\cdot x},$$

so that

$$(\mathrm{i}\gamma^\mu \partial_\mu - m)\big(v(p)\,\mathrm{e}^{+\mathrm{i}p\cdot x}\big) = \big(-\gamma^\mu p_\mu - m\big)v(p)\,\mathrm{e}^{+\mathrm{i}p\cdot x}.$$

Again using the fact that the exponential is nonzero, this gives

$$(\gamma^\mu p_\mu + m)v(p) = 0,$$

or equivalently

$$(\not{p} + m)v(p) = 0.$$

Thus the constant spinors satisfy the algebraic relations

$$(\not{p} - m)u(p) = 0, \qquad (\not{p} + m)v(p) = 0. \tag{15}$$

The $u(p)$ spinors are associated with positive-frequency solutions, while the $v(p)$ spinors are associated with negative-frequency solutions. In relativistic quantum field theory these will later be interpreted as particle and antiparticle spinors, respectively.

9.2 Rest-frame solutions

The structure becomes especially transparent in the rest frame $\mathbf{p} = 0$, where $p^\mu = (m, 0, 0, 0)$. Then the equation $(\not{p} - m)u = 0$ becomes

$$(\gamma^0 - \mathbf{1})u = 0,$$

which leaves two independent solutions, corresponding to the two spin states. Likewise $(\not{p} + m)v = 0$ yields two independent negative-frequency solutions. Thus even before introducing explicit matrices in detail, one sees that a free Dirac field carries four independent plane-wave solutions: two positive-frequency and two negative-frequency.

9.3 Explicit spinors in the Dirac representation

Up to this point many formulas were written in a chiral basis because chirality is especially transparent there. For explicit free-particle spinors, however, it is often convenient to work in the Dirac representation, where

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}.$$

With our mostly-minus convention,

$$\not{p} \equiv \gamma^\mu p_\mu = E\gamma^0 - \gamma^i p_i.$$

Let

$$u(p) = \begin{pmatrix} \phi \\ \chi \end{pmatrix},$$

where ϕ and χ are two-component spinors. Then the equation

$$(\not{p} - m)u(p) = 0$$

becomes

$$\begin{pmatrix} (E-m)\mathbf{1} & -\sigma\cdot\mathbf{p} \\ \sigma\cdot\mathbf{p} & -(E+m)\mathbf{1} \end{pmatrix} \begin{pmatrix} \phi \\ \chi \end{pmatrix} = 0,$$

which gives the coupled equations

$$(E-m)\phi - (\sigma\cdot\mathbf{p})\chi = 0, \qquad (\sigma\cdot\mathbf{p})\phi - (E+m)\chi = 0.$$

Choosing

$$\phi = \sqrt{E+m}\,\xi^{(s)},$$

with $\xi^{(s)}$ a two-component basis spinor, the second equation gives

$$\chi = \frac{\sigma\cdot\mathbf{p}}{\sqrt{E+m}}\,\xi^{(s)}.$$

Hence

$$u^{(s)}(p) = \begin{pmatrix} \sqrt{E+m}\,\xi^{(s)} \\ \dfrac{\sigma\cdot\mathbf{p}}{\sqrt{E+m}}\,\xi^{(s)} \end{pmatrix}, \qquad s = 1, 2. \tag{16}$$

Similarly, let

$$v(p) = \begin{pmatrix} \phi \\ \chi \end{pmatrix}.$$

Then the equation

$$(\not{p}+m)v(p) = 0$$

becomes

$$\begin{pmatrix} (E+m)\mathbf{1} & -\sigma\cdot\mathbf{p} \\ \sigma\cdot\mathbf{p} & -(E-m)\mathbf{1} \end{pmatrix} \begin{pmatrix} \phi \\ \chi \end{pmatrix} = 0,$$

so that

$$(E+m)\phi - (\sigma\cdot\mathbf{p})\chi = 0, \qquad (\sigma\cdot\mathbf{p})\phi - (E-m)\chi = 0.$$

Choosing

$$\chi = \sqrt{E+m}\,\eta^{(s)},$$

one finds

$$\phi = \frac{\sigma\cdot\mathbf{p}}{\sqrt{E+m}}\,\eta^{(s)}.$$

Therefore

$$v^{(s)}(p) = \begin{pmatrix} \dfrac{\sigma\cdot\mathbf{p}}{\sqrt{E+m}}\,\eta^{(s)} \\ \sqrt{E+m}\,\eta^{(s)} \end{pmatrix}, \qquad s = 1, 2. \tag{17}$$

Here $\xi^{(s)}$ and $\eta^{(s)}$ are two-component basis spinors, often chosen as eigenstates of spin along a convenient axis.

Different normalisation conventions are used in the literature. A common one is

$$\bar{u}^{(r)}(p)u^{(s)}(p) = 2m\,\delta^{rs}, \qquad \bar{v}^{(r)}(p)v^{(s)}(p) = -2m\,\delta^{rs}.$$

The precise normalisation is less important at this stage than the conceptual message: the u and v spinors solve different algebraic equations and will later acquire the interpretation of particle and antiparticle spinors.

9.4 Completeness relations

The free spinors satisfy useful completeness relations,

$$\sum_s u^{(s)}(p)\bar{u}^{(s)}(p) = \not{p}+m, \qquad \sum_s v^{(s)}(p)\bar{v}^{(s)}(p) = \not{p}-m. \tag{18}$$

These relations are important in scattering calculations and in proving many formal identities. More conceptually, they show that the u and v spinors provide a complete basis of on-shell solutions.

9.5 Why the four solutions are not redundant

One might worry that four independent solutions are “too many” for a spin-$\frac{1}{2}$ particle. The resolution is that a relativistic theory naturally describes both particle and antiparticle excitations, each with two spin states. This is already visible in the solution space before the full field-theoretic interpretation is introduced.

Remark 9.1: Single-particle language versus field language

In a purely single-particle interpretation, the negative-frequency solutions are troublesome. In quantum field theory they are not troublesome at all: they become the antiparticle modes of the field. This is one of the places where relativistic quantum mechanics points beyond itself and toward quantum field theory.

10 Spin, helicity, and chirality

10.1 Spin in relativistic theory

For a massive particle, spin is best defined in the rest frame, where the little group is the rotation group. One may then speak of spin-up and spin-down states along a chosen axis. However, once momentum is not zero, a fixed spatial axis is often not the most natural quantity to refer to. The quantity that is especially useful for relativistic fermions is helicity.

10.2 Helicity

Helicity is the projection of spin along the direction of motion:

$$h = \frac{\mathbf{S} \cdot \mathbf{p}}{|\mathbf{p}|}. \tag{19}$$

For spin-$\frac{1}{2}$ particles the eigenvalues are $\pm\frac{1}{2}$. These are called right-handed and left-handed helicity states.

The important thing to remember is that for a *massive* particle, helicity is not Lorentz invariant. One can always boost to a frame that overtakes the particle and reverses the direction of momentum, while the spin direction need not reverse in the same way. Therefore a positive-helicity massive fermion in one frame can appear as negative-helicity in another.

For a *massless* particle, however, no inertial observer can overtake the particle. In that case helicity becomes Lorentz invariant. This is why helicity plays such a central role in high-energy and massless limits.

Example 10.1: Why helicity changes for massive particles

Imagine a massive fermion moving to the right with spin aligned to the right. Its helicity is positive. An observer moving faster to the right than the particle sees the particle move to the left, while the spin direction along the chosen physical axis can remain the same. The sign of $\mathbf{S} \cdot \mathbf{p}$ flips. Helicity is therefore frame dependent for massive particles.

10.3 Chirality

Chirality is defined through γ^5, not through the classical picture of spin along momentum. A chiral eigenstate satisfies

$$\gamma^5 \psi_R = +\psi_R, \qquad \gamma^5 \psi_L = -\psi_L.$$

Using the projectors (8), any Dirac spinor decomposes into left- and right-chiral components,

$$\psi = \psi_L + \psi_R.$$

It is helpful to say explicitly that the words "left-handed" and "right-handed" are overloaded in the literature. Sometimes they refer to chirality, sometimes to helicity. In a precise field-theory discussion, left- and right-chiral mean projection by P_L and P_R.

Unlike helicity, chirality is a Lorentz-covariant algebraic property of the spinor field. In particular, the separation into left- and right-chiral components is frame independent. This is why chirality, rather than helicity, is the language in which the Standard Model fermion sector is written.

10.4 When helicity and chirality coincide

For a massless fermion, helicity and chirality line up. In that limit the Dirac equation decouples into independent equations for left- and right-chiral Weyl spinors, and one may identify left-chiral states with one helicity and right-chiral states with the opposite helicity.

For a massive fermion, they do *not* coincide in general. A massive Dirac spinor of definite helicity contains both chiralities. This is easy to understand physically: the mass term couples left- and right-chiral components. That coupling is precisely what makes a massive fermion different from a pair of independent massless Weyl fermions.

10.5 Why the distinction matters for the Standard Model

This is not a fine point. The weak interaction of the Standard Model is chiral: it couples only to left-handed fermion fields (more precisely, to left-chiral fermions and right-chiral antifermions in the gauge basis). Therefore one must be absolutely clear about the difference between chirality and helicity.

At very high energies, where $E \gg m$, helicity and chirality approximately align, and the distinction becomes less visible experimentally. But conceptually the distinction remains fundamental. Standard Model gauge interactions are written in terms of chirality projectors, not in terms of the frame-dependent classical notion of helicity.

Take-home message

Helicity answers the question: "Is the spin aligned or anti-aligned with the momentum?" Chirality answers the question: "Which eigencomponent of γ^5 am I projecting out?" For massless fermions these two notions coincide. For massive fermions they do not. The Standard Model is organised by chirality, not by helicity.

11 Antiparticles and the physical interpretation of relativistic fermion fields

11.1 The negative-energy problem

The relativistic dispersion relation has two branches,

$$E = \pm\sqrt{\mathbf{p}^2 + m^2}.$$

The Dirac equation therefore has both positive- and negative-frequency plane-wave solutions. In early attempts at relativistic quantum mechanics, this seemed disastrous. If negative-energy states were physical states of the same particle in the usual single-particle sense, why would ordinary positive-energy particles not decay endlessly into lower and lower energy configurations?

Dirac's historical answer was the "sea" picture. Modern quantum field theory gives the cleaner answer: the negative-frequency solutions are reinterpreted as antiparticle modes.

11.2 Charge conjugation in outline

One may define a charge-conjugated spinor by

$$\psi^c = C\,\bar{\psi}^{\mathrm{T}},$$

where the charge-conjugation matrix C satisfies

$$C\gamma^\mu C^{-1} = -(\gamma^\mu)^{\mathrm{T}}.$$

The precise matrix form depends on representation, but the conceptual role is clear: charge conjugation relates particle and antiparticle degrees of freedom. If ψ solves the Dirac equation in the presence of a background electromagnetic field with charge q, then ψ^c solves the corresponding equation with charge $-q$.

11.3 Particle and antiparticle interpretation of the u and v spinors

From the plane-wave analysis above, we have learned that the free Dirac equation admits four independent on-shell solutions: two $u^{(s)}(p)\,\mathrm{e}^{-\mathrm{i}p\cdot x}$ modes and two $v^{(s)}(p)\,\mathrm{e}^{+\mathrm{i}p\cdot x}$ modes, corresponding to the two spin states of the positive- and negative-frequency branches. A general free solution is therefore a superposition of all these independent plane-wave modes.

In relativistic quantum field theory one promotes the coefficients of this superposition from ordinary complex amplitudes to operators. The free Dirac field is then expanded as

$$\psi(x) = \sum_s \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_s(\mathbf{p})\, u^{(s)}(p)\, \mathrm{e}^{-\mathrm{i}p\cdot x} + b_s^\dagger(\mathbf{p})\, v^{(s)}(p)\, \mathrm{e}^{+\mathrm{i}p\cdot x} \right).$$

Here $a_s(\mathbf{p})$ annihilates a particle of momentum $\mathbf{p}$ and spin label s, while $b_s^\dagger(\mathbf{p})$ creates an antiparticle of momentum $\mathbf{p}$ and spin label s. The factor $1/\sqrt{2E_{\mathbf{p}}}$ is a conventional relativistic normalisation.

Thus the coefficients are no longer ordinary numbers; they become annihilation and creation operators. Then

- u spinors accompany annihilation of particles and creation of particles in the conjugate field,

- v spinors accompany creation of antiparticles and annihilation of antiparticles in the conjugate field.

The negative-frequency modes are therefore not pathological. They are required for a consistent local relativistic quantum theory of charged fermions.

11.4 Why antiparticles emerge naturally

The emergence of antiparticles is not something imported by hand after solving the equation. It is already encoded in the structure of relativistic quantum theory itself.

1. The relativistic energy-momentum relation has both signs for the energy.
2. The Dirac equation therefore has both positive- and negative-frequency solutions.
3. A stable and local quantum theory cannot interpret the negative-frequency branch as merely "unphysical" and forget about it.
4. Quantising the field reinterprets those modes as antiparticle excitations with positive energy.

This is one of the most conceptually striking differences between non-relativistic quantum mechanics and relativistic quantum field theory.

11.5 Why relativistic quantum theory points toward field theory

The Dirac equation can be written and studied at the level of relativistic wave mechanics, but its deepest interpretation is field theoretic. Three clues point in that direction.

- The existence of negative-frequency solutions suggests that the single-particle picture is incomplete.
- Relativistic interactions can create and annihilate particle-antiparticle pairs.
- Locality and causality are implemented most naturally in terms of fields rather than fixed-particle-number wavefunctions.

Thus Module 2 is already preparing the conceptual move from relativistic wave equations to relativistic quantum fields. This is exactly the move needed before the Standard Model Lagrangian can be written sensibly.

Remark 11.1: Particles versus fields

In collider language we often speak as if particles are the basic objects and fields are just convenient formulas. In local quantum field theory the opposite is conceptually cleaner: the fundamental objects are fields, and particles are the quantised excitations of those fields. The Dirac field is the relativistic field whose quanta are fermions and antifermions of spin $\frac{1}{2}$.

12 Lorentz-covariant bilinears and invariant fermion structures

12.1 Why bilinears matter

Module 1 already emphasised that a Lagrangian is not an arbitrary collection of terms. It must be built from invariant combinations of fields and derivatives. For fermions this immediately raises a practical question: given a spinor ψ, what combinations of ψ and $\bar{\psi}$ transform as Lorentz scalars, vectors, and other covariant objects?

The answer is expressed through fermion bilinears. These are the building blocks from which kinetic terms, mass terms, currents, and many interaction terms are constructed.

12.2 The five standard bilinear types

For a Dirac spinor one commonly considers the following bilinears:

$$\begin{aligned} &\bar{\psi}\psi && \text{scalar,} \\ &\bar{\psi}\mathrm{i}\gamma^5\psi && \text{pseudoscalar,} \\ &\bar{\psi}\gamma^\mu\psi && \text{vector,} \\ &\bar{\psi}\gamma^\mu\gamma^5\psi && \text{axial vector,} \\ &\bar{\psi}\sigma^{\mu\nu}\psi && \text{antisymmetric tensor,} \end{aligned}$$

where it is standard to define

$$\sigma^{\mu\nu} \equiv \frac{\mathrm{i}}{2}[\gamma^\mu, \gamma^\nu] = 2\Sigma^{\mu\nu}.$$

Their names indicate how they transform under Lorentz symmetry and, for the pseudo-types, how they behave under parity. The tensor bilinear is less common in elementary introductions than the scalar and vector cases, but it is part of the standard Lorentz-covariant classification and appears in magnetic-moment and dipole-type structures.

Definition 12.1: Lorentz scalar bilinear

A bilinear is Lorentz invariant if the transformation of the spinor and the Dirac adjoint combine so that the full expression is unchanged under Lorentz transformations. The simplest example is

$$\bar{\psi}\psi,$$

which is a Lorentz scalar and therefore an admissible candidate for a mass term in the Lagrangian.

12.3 Current bilinears

The vector bilinear

$$j^\mu = \bar{\psi}\gamma^\mu\psi$$

is the conserved Dirac current for the free theory. The axial current

$$j_5^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$$

plays a central role in chiral symmetry, weak interactions, and later discussions of anomalies and the chiral structure of the Standard Model.

The scalar and pseudoscalar bilinears are also important because they help classify interaction terms. For example, a Yukawa-type interaction between a scalar field ϕ and a Dirac fermion may involve $\phi\bar{\psi}\psi$ or $\phi\bar{\psi}\mathrm{i}\gamma^5\psi$ depending on the transformation properties of the scalar sector.

12.4 Why Lorentz invariance constrains model building

Once one knows the transformation properties of bilinears, one sees immediately that many seemingly plausible fermionic combinations are forbidden if one insists on Lorentz invariance. For instance, the term

$$\psi^\dagger\psi$$

is not a Lorentz scalar by itself, while

$$\bar{\psi}\psi$$

is. Likewise, one cannot simply couple an arbitrary spinor component to an arbitrary bosonic quantity and expect covariance to survive.

This is an early but essential lesson in field theory: symmetry requirements do real work. They eliminate many candidate terms and sharply restrict what a consistent relativistic theory may contain.

Example 12.1: A schematic comparison

The expression

$$\mathcal{L}_1 = -m\,\bar{\psi}\psi$$

is Lorentz invariant and is the standard Dirac mass term. By contrast,

$$\mathcal{L}_2 = -m\,\psi^\dagger\psi$$

looks innocent but is not written in a Lorentz-invariant way. The distinction is entirely due to transformation properties.

13 Fermion mass terms and relativistic invariance

13.1 The Dirac mass term

The Lorentz-invariant mass term for a Dirac fermion is

$$\mathcal{L}_{\text{mass}} = -m\,\bar{\psi}\psi. \tag{20}$$

In terms of chiral components,

$$\bar{\psi}\psi = \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L.$$

A useful detail, often skipped too quickly, is that same-chirality terms vanish:

$$\bar{\psi}_L\psi_L = \bar{\psi}P_RP_L\psi = 0, \qquad \bar{\psi}_R\psi_R = \bar{\psi}P_LP_R\psi = 0.$$

So the Dirac mass term necessarily couples opposite chiralities. This is crucial: a massive Dirac fermion cannot consist of a single Weyl spinor alone.

13.2 Why the mass term mixes chiralities

Because γ^5 anticommutes with γ^μ, the chiral projections are interchanged by the Dirac operator. In physical language, the mass term ties together the two chiral sectors. In the massless limit the two sectors decouple and the theory splits into independent Weyl equations.

The contrast with the kinetic term is instructive. Using (9), one finds schematically

$$\bar{\psi}\,\mathrm{i}\gamma^\mu\partial_\mu\psi = \bar{\psi}_L\,\mathrm{i}\gamma^\mu\partial_\mu\psi_L + \bar{\psi}_R\,\mathrm{i}\gamma^\mu\partial_\mu\psi_R,$$

whereas the mass term is off-diagonal in chirality:

$$\bar{\psi}\psi = \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L.$$

That is why setting $m = 0$ increases the symmetry of the free theory: left- and right-chiral sectors propagate independently.

This immediately prepares one for one of the core structural facts of the Standard Model: left- and right-handed fermions do not transform the same way under the weak gauge group. Therefore ordinary fermion masses are not inserted by hand in the electroweak-symmetric theory. They arise only after the Higgs mechanism makes it possible to connect the two chiral sectors in a gauge-invariant manner.

13.3 A short comment on Weyl and Majorana mass structures

For a purely left-handed Weyl spinor, a Lorentz-invariant mass term requires a more delicate contraction of spinor indices and, depending on the gauge quantum numbers, may or may not be allowed. In many contexts this leads to Majorana-type masses. The present module does not need the full formal development, but one conceptual point is worth recording: Lorentz invariance is necessary for a fermion mass term, yet not sufficient in the Standard Model. One must also respect gauge symmetry. That bridge will become central in later modules.

13.4 Kinetic term and full free Dirac Lagrangian

The free Dirac Lagrangian is

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi. \tag{21}$$

The first term is the kinetic term. It is Lorentz invariant because $\bar{\psi}\gamma^\mu\psi$ is a vector and ∂_μ is a covector, so the contraction is a scalar. The second term is the mass term discussed above. Writing the kinetic term in chiral pieces makes the structure especially transparent:

$$\bar{\psi}\,\mathrm{i}\gamma^\mu\partial_\mu\psi = \bar{\psi}_L\,\mathrm{i}\gamma^\mu\partial_\mu\psi_L + \bar{\psi}_R\,\mathrm{i}\gamma^\mu\partial_\mu\psi_R.$$

So the kinetic term respects the separation into left- and right-chiral sectors, while the mass term mixes them. This is exactly the pattern that later becomes central in the Standard Model, where the kinetic terms can be made gauge covariant for each chiral multiplet separately, but a naive mass term is not automatically compatible with electroweak gauge symmetry.

Thus both the dynamics and the mass structure of the free theory are already controlled by Lorentz representation theory.

Take-home message
For fermions, Lorentz invariance is not a vague background requirement. It tells us what the field is, what the admissible kinetic term is, what the admissible mass term is, and what kinds of currents and interactions are possible. The later Standard Model Lagrangian rests directly on this logic.

14 From relativistic fermions to the Standard Model

14.1 Why the Standard Model is built from relativistic fermion fields

The quarks and leptons of the Standard Model are not added to the theory as classical particles. They appear as relativistic fermion fields. Their free propagation is governed by the Dirac or Weyl structure developed above, and their interactions are introduced by replacing ordinary derivatives with covariant derivatives once gauge symmetry is imposed.

One already meets the seed of this transition in the free Dirac theory itself. The Lagrangian (21) has a global phase symmetry

$$\psi \to e^{iq\alpha}\psi, \qquad \alpha = \text{constant},$$

with conserved current $j^\mu = \bar{\psi}\gamma^\mu\psi$. Module 3 will ask what happens when the phase parameter is promoted to a spacetime function, $\alpha \to \alpha(x)$. At that point the ordinary derivative no longer transforms covariantly, and the gauge field becomes necessary. So the bridge to gauge theory is not external to Module 2. It is already latent in the relativistic fermion Lagrangian.

At this point, one can already see the architecture of the later theory:

- spacetime symmetry tells us that matter must be described by relativistic spinor fields,
- Lorentz invariance tells us how to write kinetic and mass structures,
- internal gauge symmetry will tell us which derivatives and interaction terms are allowed,
- chirality will determine how weak interactions distinguish left and right components.

14.2 Why left- and right-handed components matter so much

In the Standard Model, left-handed quarks and leptons are assembled into $\mathrm{SU}(2)_L$ doublets, while right-handed components are $\mathrm{SU}(2)_L$ singlets. This is why Module 2 is not merely a chapter on "relativistic quantum mechanics". It provides the language in which the chiral weak interaction can even be stated. Without the distinction between Weyl components and Dirac spinors, the gauge structure of the Standard Model would look unmotivated and mysterious.

This also explains why fermion masses become conceptually subtle. The electroweak theory treats left- and right-chiral fields differently, but a Dirac mass term couples them. Therefore the naive free-theory mass term cannot simply be copied unchanged into the gauge-symmetric Standard Model. Later modules will show how the Higgs field and Yukawa couplings solve this problem in a gauge-invariant way.

14.3 Relativistic invariance as a structural constraint

One of the strongest messages of this module is that Lorentz symmetry constrains the construction of particle-physics models at every step. The field content must belong to representations of the Lorentz group. The terms in the Lagrangian must be Lorentz scalars. Mass terms tie together specific chiral structures. Currents and interactions must be built from covariant bilinears. The Standard Model is therefore not merely a gauge theory. It is a gauge theory built on top of a tightly constrained relativistic spinor framework.

14.4 Bridge to Module 3

Module 3 will develop gauge symmetry and the construction of the Standard Model. The transition is conceptually natural. The key move is very specific: starting from the free relativistic fermion theory,

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}(\mathrm{i}\gamma^\mu \partial_\mu - m)\psi,$$

one first notes its global internal phase symmetry and then asks what changes when that symmetry is made local. The answer is that

$$\partial_\mu \longrightarrow D_\mu,$$

and the gauge field is introduced precisely so that $D_\mu\psi$ transforms covariantly. In other words, Module 2 supplies the relativistic fermion field to which Module 3 applies the gauge principle.

Module	**Conceptual contribution**
Module 1	States, operators, symmetry, groups, generators, representations, invariants, and the global-to-local transition.
Module 2	Lorentz covariance, spacetime representations, spinors, gamma matrices, Dirac fields, chirality, and relativistic fermion bilinears.
Module 3	Gauge principle, covariant derivative, Abelian and non-Abelian gauge theories, and the full Standard Model gauge structure.

The central conceptual bridge is this: Module 2 tells us *what relativistic fermions are*. Module 3 tells us *how those fermions interact through gauge symmetry*.

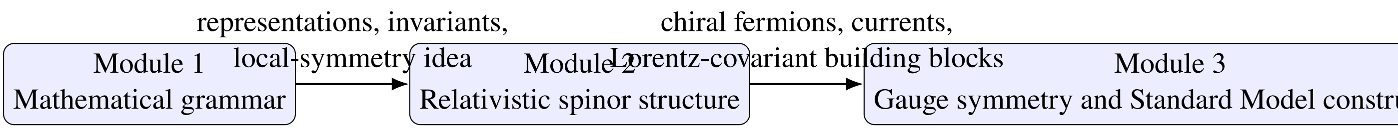

Figure 2: Conceptual bridge from Module 1 through Module 2 to Module 3. Module 2 supplies the relativistic field content and chiral structure needed before gauge-theory construction can be made concrete.

15 Common conceptual confusions and how to avoid them

Because this module introduces several closely related but distinct ideas, it is useful to end the main discussion with an explicit list of common confusions.

15.1 Lorentz transformation versus coordinate change

A Lorentz transformation can be viewed actively or passively. In the passive viewpoint one changes coordinates between inertial frames. In the active viewpoint one keeps coordinates fixed and transforms the physical object. Both descriptions are legitimate, but one must not mix them casually inside a derivation. The transformed coordinates and the transformed field must be handled consistently.

15.2 Vector versus spinor

A spinor is not a short vector and not a peculiar four-vector. It belongs to a different representation of spacetime symmetry. The most visible signal is the behaviour under rotations: spinors require a 4π rotation to return to themselves, whereas vectors return after 2π.

15.3 Weyl versus Dirac spinor

A Weyl spinor is an irreducible chiral representation. A Dirac spinor is the direct sum of left- and right-chiral Weyl spinors. Confusing the two leads quickly to confusion about masses and weak interactions. In particular, a single Weyl spinor does not contain the same degrees of freedom as a massive Dirac fermion.

15.4 Helicity versus chirality

These notions are often conflated because they coincide for massless fermions. But they are conceptually different. Helicity is the projection of spin on momentum and depends on the observer for massive particles. Chirality is the eigenvalue of γ^5 and is defined by projection operators. The Standard Model is built in terms of chirality.

15.5 Particle versus field

A particle is an excitation observed in experiments. A field is the underlying relativistic quantum object whose quantised excitations appear as particles. The Dirac equation is often introduced as a wave equation for a relativistic particle, but its deepest role is as the field equation for the Dirac field.

15.6 Negative-energy solution versus antiparticle

The negative-frequency solutions of the Dirac equation are not simply discarded branches. In quantum field theory they are reinterpreted as antiparticle modes with positive energy. Calling them “negative-energy particles” after quantisation is misleading.

15.7 Invariant quantity versus covariant object

A covariant object transforms in a definite way under Lorentz transformations. An invariant quantity does not change at all. A four-vector is covariant, not invariant. Its contraction with another four-vector may be invariant. Keeping this distinction clear helps enormously when building Lagrangians.

15.8 An equation that is relativistic versus one that merely looks relativistic

An equation written with spacetime indices is not automatically Lorentz covariant. One must know how every object transforms. For example, $\psi^\dagger\psi$ looks superficially similar to a scalar density, but by itself it is not the invariant fermion bilinear that appears in the relativistic Lagrangian. Covariance is a structural statement, not a typographical one.

Remark 15.1: Why this list matters

Each of these confusions appears repeatedly when students first meet relativistic fermion theory. Writing them down is not pedagogical over-caution. It is part of making the subject learnable. Most later confusion in gauge theory, chirality, and Standard Model fermion structure can be traced back to one or more of the distinctions listed above.

16 Summary and take-home messages

These notes have expanded the logic of the AGH Module 2 description into a self-contained text. The central question of the module has been: *what is the minimal relativistic language needed to describe spin-$\frac{1}{2}$ particles consistently and to prepare for the field-theoretic formulation of the Standard Model?*

The answer can be summarised in several linked points.

1. Particle physics requires Lorentz symmetry because it is a relativistic theory of high-energy phenomena.
2. Minkowski spacetime, four-vectors, invariant intervals, and covariant contractions provide the geometric language in which that symmetry is expressed.
3. Fields must transform in representations of the Lorentz group. Scalars and vectors are not enough to describe spin-$\frac{1}{2}$ matter.
4. Spinors are therefore required. The basic relativistic spinor structures are Weyl spinors, while Dirac spinors combine left- and right-chiral pieces.
5. The Dirac equation is the first-order relativistic wave equation that consistently describes free spin-$\frac{1}{2}$ fermions and reproduces the relativistic mass-shell condition.
6. Gamma matrices and the Clifford algebra are not formal tricks; they are the algebraic machinery that makes the Dirac equation possible.
7. Positive- and negative-frequency solutions of the Dirac equation foreshadow particle and antiparticle degrees of freedom. A full interpretation naturally leads to quantum field theory.
8. Helicity and chirality are distinct concepts. Their difference is essential for understanding the Standard Model.
9. Lorentz-covariant bilinears determine the admissible fermion kinetic terms, mass terms, and current structures.
10. Module 2 therefore prepares the exact language needed for Module 3, where gauge symmetry and the Standard Model fermion sector will be constructed explicitly.

Take-home message
Module 1 supplied the grammar of symmetry, representations, and invariants. Module 2 supplied the relativistic spacetime representations needed for matter fields. The next step is then natural: gauge symmetry acts on these relativistic fields, and the Standard Model emerges as a tightly constrained relativistic quantum field theory of chiral fermions, gauge bosons, and the Higgs field.

A Compact notation and formula sheet for Module 2

Spacetime conventions

Symbol	Meaning
$\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$	Minkowski metric (mostly-minus convention)
$x^\mu = (t, \mathbf{x})$	spacetime four-vector
$p^\mu = (E, \mathbf{p})$	energy-momentum four-vector
$x_\mu = \eta_{\mu\nu} x^\nu$	lowered-index four-vector
$p^\mu p_\mu = E^2 - \mathbf{p}^2$	invariant norm of four-momentum
$p^\mu p_\mu = m^2$	on-shell mass-shell relation
$\Box = \partial^\mu \partial_\mu$	d'Alembert operator

Lorentz symmetry

$$
\begin{aligned}
x'^\mu &= \Lambda^\mu{}_\nu x^\nu, \\
\Lambda^{\mathrm{T}} \eta \Lambda &= \eta, \\
\omega_{\mu\nu} &= -\omega_{\nu\mu}, \\
[M^{\mu\nu}, M^{\rho\sigma}] &= \mathrm{i}\left(\eta^{\nu\rho} M^{\mu\sigma} - \eta^{\mu\rho} M^{\nu\sigma} - \eta^{\nu\sigma} M^{\mu\rho} + \eta^{\mu\sigma} M^{\nu\rho}\right), \\
J^i &= \frac{1}{2}\epsilon^{ijk} M^{jk}, \qquad K^i = M^{0i}, \\
\left[J^i, J^j\right] &= \mathrm{i}\epsilon^{ijk} J^k, \qquad \left[J^i, K^j\right] = \mathrm{i}\epsilon^{ijk} K^k, \qquad \left[K^i, K^j\right] = -\mathrm{i}\epsilon^{ijk} J^k.
\end{aligned}
$$

Gamma matrices and spinor structure

$$
\begin{aligned}
\{\gamma^\mu, \gamma^\nu\} &= 2\eta^{\mu\nu} \mathbf{1}, \\
\bar{\psi} &= \psi^\dagger \gamma^0, \\
\gamma^5 &= \mathrm{i}\gamma^0 \gamma^1 \gamma^2 \gamma^3, \\
P_L &= \frac{1}{2}(1 - \gamma^5), \qquad P_R = \frac{1}{2}(1 + \gamma^5), \\
\psi_L &= P_L \psi, \qquad \psi_R = P_R \psi, \\
\Sigma^{\mu\nu} &= \frac{\mathrm{i}}{4}\left[\gamma^\mu, \gamma^\nu\right], \\
\gamma^\mu P_L &= P_R \gamma^\mu, \qquad \gamma^\mu P_R = P_L \gamma^\mu.
\end{aligned}
$$

Dirac equation and free-fermion structures

$$\begin{aligned}
(\mathrm{i}\gamma^\mu \partial_\mu - m)\psi &= 0,\\
\bar{\psi}(\mathrm{i}\overleftarrow{\partial}_\mu \gamma^\mu + m) &= 0,\\
j^\mu &= \bar{\psi}\gamma^\mu\psi,\\
\partial_\mu j^\mu &= 0,\\
(\not{p} - m)u(p) &= 0,\\
(\not{p} + m)v(p) &= 0,\\
\sum_s u^{(s)}(p)\bar{u}^{(s)}(p) &= \not{p} + m,\\
\sum_s v^{(s)}(p)\bar{v}^{(s)}(p) &= \not{p} - m.
\end{aligned}$$

Helicity and chirality reminder

Concept	**Meaning**
Helicity	projection of spin along momentum: $h = \mathbf{S} \cdot \mathbf{p}/|\mathbf{p}|$
Chirality	eigenvalue under γ^5 or equivalently projection by P_L, P_R
Massless limit	helicity and chirality coincide
Massive case	helicity is frame dependent; chirality is the relevant field-theoretic label
Standard Model lesson	weak interactions are chiral

Useful fermion bilinears

$$\begin{aligned}
\bar{\psi}\psi &\quad \text{scalar},\\
\bar{\psi}\mathrm{i}\gamma^5\psi &\quad \text{pseudoscalar},\\
\bar{\psi}\gamma^\mu\psi &\quad \text{vector},\\
\bar{\psi}\gamma^\mu\gamma^5\psi &\quad \text{axial vector},\\
\bar{\psi}\sigma^{\mu\nu}\psi &\quad \text{antisymmetric tensor},\\
\sigma^{\mu\nu} &= \frac{\mathrm{i}}{2}[\gamma^\mu, \gamma^\nu] = 2\Sigma^{\mu\nu}.
\end{aligned}$$

B Suggested references for further study

- Mark Thomson, *Modern Particle Physics*, especially the chapters on special relativity, the Dirac equation, helicity, chirality, and weak interactions.
- David Tong, *The Standard Model*, especially the sections on spacetime symmetries, Lorentz representations, and spinors.

- Fernando Quevedo, *The Standard Model*, especially Chapter 2 on spacetime symmetries, spinors, and the route from Lorentz symmetry to Standard Model field content.
- Standard references on relativistic quantum mechanics and quantum field theory for a deeper treatment of Lorentz group representation theory, charge conjugation, and field quantisation.

About these notes

These notes are intended as a self-contained pedagogical introduction to relativistic symmetries and fermions in the context of the Standard Model. Their structure and emphasis reflect the needs of an introductory MSc-level treatment, with particular focus on Lorentz symmetry, spinor representations, the Dirac equation, chirality, and the bridge to gauge theory. The exposition has benefited from standard lecture-note and textbook treatments of the subject, including works by Mark Thomson, David Tong, and Fernando Quevedo. The presentation, wording, notation choices, and figures in the present document have been adapted and organised for the purposes of these lecture notes.