AGH University of Krakow

Faculty of Physics and Applied Computer Science

MODULE 4

QUANTUM CHROMODYNAMICS AND THE STRONG INTERACTION

Supporting Lecture Notes for *The Standard Model*

The Guideline

The aim of this expanded module note is to make visible how the non-Abelian gauge structure of $\mathrm{SU}(3)_c$ *becomes the physics of the strong interaction. Starting from quark and gluon fields, the covariant derivative, the non-Abelian field-strength tensor, and the QCD Lagrangian, the note develops the central physical consequences of gluon self-interactions: the running of the strong coupling, asymptotic freedom at short distances, infrared strong coupling, confinement, and the emergence of hadrons as color-singlet states. The second half of the note then turns to the phenomenological face of QCD: the parton model, deep inelastic scattering, parton distribution functions, hadronization, jets, and selected collider applications. Throughout the note, the emphasis is on seeing QCD both as a mathematically constrained Yang-Mills theory and as the physical origin of proton structure and strong-interaction signatures in experiment.*

Prepared for the Standard Model course

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June 8, 2026

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Preface

These expanded supporting notes are written for *Module 4: Quantum Chromodynamics (QCD) and the Strong Interaction*. The pedagogical goal is not only to present the main formulas of QCD, but to make visible the constructive chain that connects the formal Yang–Mills structure of the strong interaction to the experimentally observed hadronic world. In that sense, the note is organized around a broad conceptual flow:

non-Abelian gauge symmetry $\longrightarrow$ QCD fields and Lagrangian $\longrightarrow$ running coupling $\longrightarrow$ confinement and hadrons $\longrightarrow$ partons, DIS, and jets, and collider applications.

The note naturally continues the logic of Module 3. There, the general gauge principle was developed from global internal symmetry to local symmetry, covariant derivatives, gauge fields, field strengths, and gauge-invariant Lagrangians. Here, that general framework is specialized to the first full non-Abelian sector of the Standard Model, namely the $\mathrm{SU}(3)_c$ gauge theory of quarks and gluons. For this reason, the present module does not re-derive the gauge principle from scratch. Instead, it starts from the Yang–Mills structure already established and studies its strong-interaction realization in detail.

The note is intended to be self-contained at the level of a serious MSc student. It assumes familiarity with relativistic notation, Dirac fields, basic Lie-algebra language, and the gauge-theory viewpoint developed in the preceding module. Several short worked examples, take-home summaries, and guided checks are included to help the reader test whether each conceptual step has really been understood. At the same time, the note is designed to make clear how the formal structure of QCD becomes visible in real high-energy processes: in the partonic structure of the proton, in deep inelastic scattering, in hadronization and jets, and in representative proton–proton collision processes at the LHC. In particular, the later sections highlight how QCD enters inclusive jet and dijet production, multijet final states, heavy-quark production, top-quark pair production, Drell–Yan processes accompanied by QCD radiation, and Higgs-boson production through gluon fusion.

It is equally important to state the boundaries of the module. These notes develop the formal QCD framework, the running of the strong coupling, asymptotic freedom, infrared strong coupling, confinement, hadrons as color-singlet states, the parton model, deep inelastic scattering, PDFs, hadronization, jets, and selected collider applications. They therefore aim to show not only how QCD explains the existence of hadrons and proton structure, but also how it shapes the dominant event environment of modern hadron colliders and controls major Standard Model production channels in proton–proton collisions. They do *not* attempt a full advanced treatment of perturbative QCD, a detailed renormalization proof of the beta function, a full lattice-QCD review, or a replacement for later modules on electroweak symmetry breaking and general amplitudes/cross sections.

Conventions and notation

- Metric signature: $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.
- Natural units: $\hbar = c = 1$.
- Clifford algebra: $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$.
- Fundamental color indices: $i, j, k = 1, 2, 3$.

- Adjoint color indices: $a, b, c = 1, \dots, 8$.
- Generator normalization in the fundamental representation:

$$\mathrm{Tr}(T^a T^b) = \tfrac{1}{2}\,\delta^{ab}.$$

- Lie algebra:

$$\left[T^a, T^b\right] = \mathrm{i} f^{abc} T^c.$$

- Strong coupling notation:

$$\alpha_s = \frac{g_s^2}{4\pi}.$$

- Lie-algebra-valued gluon field:

$$G_\mu(x) = G_\mu^a(x)\, T^a.$$

- QCD covariant derivative:

$$D_\mu = \partial_\mu - \mathrm{i} g_s\, G_\mu = \partial_\mu - \mathrm{i} g_s\, T^a G_\mu^a.$$

1 Introduction and module roadmap

1.1 Why QCD deserves its own module

The strong interaction is not a minor appendage to the Standard Model. It is the interaction that binds quarks into hadrons, determines the structure of protons and neutrons, and thereby shapes almost all visible matter. At the formal level, it is also the simplest fully realized non-Abelian gauge sector of the Standard Model. For these reasons, QCD deserves to be studied not merely as one item in a particle chart, but as a theory whose internal mathematical logic and physical consequences can be followed in detail.

1.2 From Module 3 to Module 4

In Module 3 the central lesson was that once an internal symmetry is promoted from global to local, the ordinary derivative fails and must be replaced by a covariant derivative. The corresponding gauge field is not an optional extra; it is required by local symmetry. This logic led from the Abelian $\mathrm{U}(1)$ case of QED to the general Yang–Mills framework. The present module now specializes that framework to the color gauge group $\mathrm{SU}(3)_c$. We therefore move from *generic non-Abelian gauge theory* to the *specific strong-interaction sector of the Standard Model*.

A useful way to phrase the transition is this. Module 3 asked: how does local symmetry force the form of an interaction? Module 4 asks: what happens when that local symmetry is the exact color gauge symmetry of nature? Part I answers that question at the level of fields, representations, and the QCD Lagrangian. Part II then turns to the experimentally visible consequences.

1.3 What this module will and will not do

This module develops:

- color as the internal charge of QCD,
- quark and gluon fields,
- the $\mathrm{SU}(3)_c$ algebra and its representations at the level needed for QCD,
- the QCD covariant derivative,
- the non-Abelian field-strength tensor,
- the full QCD Lagrangian,
- the origin of quark–gluon, triple-gluon, and quartic-gluon interaction terms,
- the running of α_s,
- asymptotic freedom and infrared strong coupling,
- confinement, hadrons, the parton model, DIS, PDFs, jets, and selected collider applications.

It does *not* attempt a full advanced treatment of loop calculations, a full DGLAP derivation, detailed detector reconstruction, or a general amplitudes/cross-sections course. Those belong mainly to later modules or to more advanced dedicated courses.

1.4 Roadmap of the note

The logic of the note can be summarized schematically as

$$\begin{aligned}\mathrm{SU}(3)_c &\longrightarrow \text{quark and gluon fields} \longrightarrow D_\mu \longrightarrow G^a_{\mu\nu}\\ &\longrightarrow \mathcal{L}_{\rm QCD} \longrightarrow \alpha_s(Q^2) \longrightarrow \text{asymptotic freedom}\\ &\longrightarrow \text{confinement} \longrightarrow \text{hadrons} \longrightarrow \text{partons} \longrightarrow \text{DIS}\\ &\longrightarrow \text{jets} \longrightarrow \text{collider applications.}\end{aligned}$$

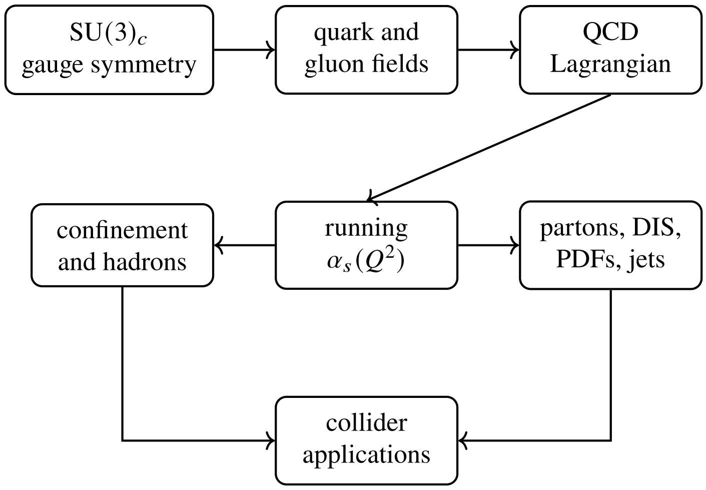


Figure 1: Conceptual roadmap of Module 4: from the non-Abelian gauge structure of SU(3)$_c$ to the observable physics of hadrons, proton structure, jets, and collider signatures.

Take-home message

In Module 4, the Yang–Mills construction from Module 3 is no longer only a formal template. It becomes the concrete theory of color, quarks, gluons, hadrons, and collider signatures.

Part I. QCD as a Non-Abelian Gauge Theory

2 Why QCD is needed in the Standard Model

2.1 Strongly interacting matter and the hadronic world

Experiment does not show us free quarks as asymptotic particles. Instead, it shows a rich hadronic world: mesons, baryons, resonances, and jets. At a deeper level, however, many regularities in that hadronic world point toward an underlying theory whose fundamental degrees of freedom are more elementary than hadrons themselves. QCD provides that theory.

The strong interaction must explain at least three broad facts:

1. hadrons are composite rather than elementary,
2. the observed hadronic spectrum reflects an internal structure not reducible to electric charge alone,

3. strongly interacting physics behaves in a way profoundly different from electromagnetism.

At this stage it is useful to keep two languages separate but connected. One language is the language of observed hadrons. The other is the language of the QCD Lagrangian, written in terms of quark and gluon fields. Part I is mainly about the second language, but it must already explain why the first language is not arbitrary. In particular, the possibility of building color-singlet states is already encoded in the formal structure of the theory.

2.2 Why electromagnetism cannot explain hadrons

Electromagnetism is an Abelian gauge theory with a single neutral gauge boson, the photon. It can explain the interaction of electrically charged particles, but it cannot account for the multiplicity of hadronic states, the existence of quarks with an additional internal charge, or the special role played by strongly interacting bound states. In particular, the internal structure required of quarks inside baryons cannot be accounted for using electric charge alone.

A second point is even more important. The gauge boson of electromagnetism does not carry electric charge, and the field-strength tensor in QED is linear in the photon field. The strong interaction, by contrast, requires a non-Abelian structure in which the gauge bosons themselves participate in the gauge charge structure.

A historically suggestive example is provided by baryons such as the Δ^{++}, whose quark-model content is uuu. Without an additional internal degree of freedom, the construction of such states quickly runs into consistency problems once one combines flavor, spin, and statistics. The introduction of color is not merely decorative; it is what allows the hadronic spectrum to be organized consistently.

2.3 The need for color as a new internal quantum number

To describe strongly interacting matter consistently, quarks must carry an additional internal label beyond flavor and electric charge. This label is called *color*. It is not a visual color, but a quantum number associated with a local gauge symmetry. In QCD, quarks come in three color states. The gauge symmetry acting on this color degree of freedom is

$$\mathrm{SU}(3)_c. \tag{2.1}$$

The subscript c reminds us that this is the color symmetry, not the approximate flavor $\mathrm{SU}(3)$ that also appears elsewhere in hadronic physics.

Definition 2.1: Color charge

Color is the internal gauge quantum number associated with the exact local symmetry group $\mathrm{SU}(3)_c$ of QCD. A colored field transforms nontrivially under this group; a color-singlet field or state is invariant under the group action.

2.4 QCD as the $\mathrm{SU}(3)_c$ sector of the Standard Model

At the level of gauge structure, the Standard Model is organized by

$$G_{\mathrm{SM}} = \mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y. \tag{2.2}$$

Among these three factors, $\mathrm{SU}(3)_c$ is the symmetry group of the strong interaction. QCD is therefore the Yang–Mills theory associated with the color factor of the Standard Model.

Remark 2.1: Why QCD is special inside the Standard Model

The conceptual role of QCD in the Standard Model is special. It is simultaneously:

- a gauge theory of fundamental colored fields, and
- the underlying theory of the hadronic world that experiments directly observe.

This is why a formal discussion of QCD cannot stay completely isolated from hadronic state construction, even in Part I.

A structural viewpoint: At this point it is useful to say explicitly what kind of theory QCD is. It is not merely a model introduced to classify hadrons after the fact. Once color is identified as an exact local internal symmetry and the gauge principle is taken seriously, the strong interaction is forced into the Yang–Mills form associated with $\mathrm{SU}(3)_c$. In that sense, QCD is both a theory of fundamental colored fields and the deeper field-theoretic explanation of why the hadronic world is organized as it is. This is why Part I may focus on fields, generators, and Lagrangians while still remaining directly relevant to the observed physics of mesons, baryons, and jets.

3 Color charge and the $\mathrm{SU}(3)_c$ symmetry

3.1 Color as an internal degree of freedom

Color is an internal quantum number labeling how a quark transforms under the local symmetry group $\mathrm{SU}(3)_c$. At a given spacetime point, a quark field may be regarded as a three-component object in color space:

$$q_f(x) = \begin{pmatrix} q_f^1(x) \\ q_f^2(x) \\ q_f^3(x) \end{pmatrix}, \tag{3.1}$$

where the flavor label f distinguishes u, d, s, c, b, t, while the superscript $i = 1, 2, 3$ distinguishes the three color components.

Under a local color transformation, the quark field transforms as

$$q_f(x) \longrightarrow q'_f(x) = U(x)\, q_f(x), \qquad U(x) \in \mathrm{SU}(3)_c. \tag{3.2}$$

Thus color is not an accidental label; it is the defining internal charge of QCD.

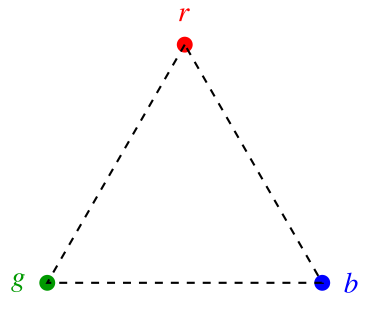
3.2 Quark fields as color triplets

The quark field transforms in the fundamental representation of $\mathrm{SU}(3)_c$, often denoted simply as the **3**. The statement that quarks are color triplets means precisely that the field carries three components mixed by 3×3 special unitary matrices.

For antiquarks the transformation law is conjugate:

$$\bar{q}_f(x) \longrightarrow \bar{q}'_f(x) = \bar{q}_f(x)\, U^{-1}(x), \tag{3.3}$$

so antiquarks transform in the conjugate representation $\bar{\mathbf{3}}$. In practical color-language one often calls this the anti-triplet representation.



$q^i(x)$ in the fundamental representation of $\mathrm{SU}(3)_c$

Figure 2: Schematic representation of the three color states of a quark field in the fundamental representation of $\mathrm{SU}(3)_c$.

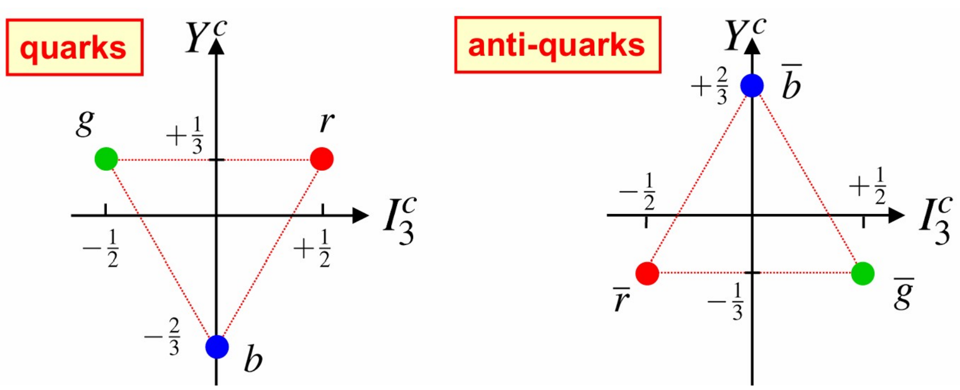


Figure 3: Color-triplet and anti-triplet state triangles, useful for visualizing the quark and antiquark color multiplets.

3.3 Why three colors

At the formal level, three colors correspond to the dimension of the fundamental representation of the gauge group. At the physical level, the existence of three colors is what allows color-singlet hadrons to be built in the observed way. In particular:

- the simplest and most important color-singlet hadronic combinations are mesons, built from a quark and an antiquark,
- and baryons, built from three quarks.

At the physical level, the existence of three colors is what allows the simplest and most important color-singlet hadronic states to be built in the observed way. In particular, mesons arise from quark–antiquark combinations, while baryons arise from three-quark combinations.

The existence of a three-dimensional color space is therefore not arbitrary decoration; it is central to the structure of hadronic states.

There is also a useful state-counting lesson. A two-quark color combination transforms as

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \bar{\mathbf{3}}, \tag{3.4}$$

so it contains no singlet. By contrast,

$$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}, \tag{3.5}$$

and

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}. \tag{3.6}$$

This already explains why $q\bar{q}$ and qqq can contain color-singlet states, while qq alone cannot.

3.4 Color versus flavor: two different $\mathrm{SU}(3)$'s

It is important not to confuse two different uses of $\mathrm{SU}(3)$ in particle physics.

- $\mathrm{SU}(3)_c$ is an exact local gauge symmetry of the Standard Model and defines QCD.
- Flavor $\mathrm{SU}(3)$ is an approximate global symmetry of hadronic physics associated with the approximate similarity of the light quarks u, d, s.

The distinction is more than terminological. Color $\mathrm{SU}(3)_c$ acts on an internal gauge label and determines the existence of gluons and the form of the QCD interaction. Flavor $\mathrm{SU}(3)$ organizes hadronic states only approximately, because the light-quark masses are not equal and electroweak effects also distinguish the flavors.

Remark 3.1: Flavor SU(3) and color SU(3) are not the same symmetry

The multiplet logic of flavor $\mathrm{SU}(3)$ is pedagogically useful because it teaches students what triplets, octets, and singlets look like in practice. But one must keep the conceptual hierarchy straight: flavor $\mathrm{SU}(3)$ is an approximate global hadronic symmetry, whereas color $\mathrm{SU}(3)_c$ is the exact local gauge symmetry of QCD. The analogy is helpful; the two symmetries are not the same physics.

3.5 Why color resolves hadronic state-construction problems

Color allows one to form physically admissible hadronic states that are invisible as free color charges. For mesons one has the decomposition

$$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}, \tag{3.7}$$

so the color-singlet meson state may be written as

$$|1\rangle_{q\bar{q}} = \frac{1}{\sqrt{3}} \left(|r\bar{r}\rangle + |g\bar{g}\rangle + |b\bar{b}\rangle \right). \tag{3.8}$$

For baryons, the three-quark singlet is built from the totally antisymmetric tensor,

$$|1\rangle_{qqq} = \frac{1}{\sqrt{6}}\left(|rgb\rangle - |rbg\rangle + |gbr\rangle - |grb\rangle + |brg\rangle - |bgr\rangle\right) \propto \epsilon_{ijk} q^i q^j q^k. \tag{3.9}$$

These formulas do more than provide mnemonic wavefunctions. They show that color is the internal structure that makes hadronic singlets possible at all. This is why color is already part of the formal QCD story, not something that should be postponed entirely to later phenomenology.

Definition 3.1: Singlet state

A singlet state of a symmetry group is a state that is invariant under the group action. For QCD this means that under $SU(3)_c$ transformations the state is unchanged. Equivalently, all generators annihilate the singlet state.

Remark 3.2: A necessary condition is not always sufficient

A color-singlet state must certainly have zero net color quantum numbers, but that condition alone is not the full definition. The stronger statement is group-theoretic invariance: a true singlet is annihilated by the generators or, equivalently, remains unchanged under the group action. In this sense, the notion of a singlet in QCD is directly analogous to the familiar idea of a singlet in angular-momentum addition: it is not merely a state with vanishing quantum-number sum, but a state that transforms trivially under the symmetry.

Example 3.1: Why qq does not form a color singlet

The tensor product of two quark triplets is

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \bar{\mathbf{3}}.$$

Because no **1** appears on the right-hand side, a two-quark state does not contain a color singlet. This is the group-theoretic reason why a diquark by itself is not an isolated hadron in the same way that a meson or baryon can be.

Guided checks

1. Verify by state counting that $\mathbf{3} \otimes \bar{\mathbf{3}}$ contains nine states, decomposed into an octet plus a singlet.
2. Explain why Eq. (3.8) is invariant under a common color rotation of the quark and antiquark.
3. Check that $\mathbf{3} \otimes \mathbf{3}$ contains no singlet and therefore cannot describe an isolated two-quark hadron.
4. Show that the color factor in Eq. (3.9) is totally antisymmetric under interchange of any two quarks.

4 Generators, algebra, and representations of SU(3)

4.1 $\mathrm{SU}(3)$ as a Lie group of local symmetry

The group $\mathrm{SU}(3)$ consists of complex 3×3 unitary matrices of determinant one:

$$U^\dagger U = 1, \qquad \det U = 1. \tag{4.1}$$

A local gauge transformation is represented by a spacetime-dependent matrix $U(x) \in \mathrm{SU}(3)_c$. For infinitesimal transformations it is convenient to write

$$U(x) \simeq 1 + i\alpha^a(x)T^a, \tag{4.2}$$

where the eight Hermitian generators T^a span the Lie algebra of $\mathrm{SU}(3)$.

4.2 Generators and the Gell–Mann matrices

A standard basis for the generators is provided by the Gell–Mann matrices λ^a, with

$$T^a = \frac{\lambda^a}{2}. \tag{4.3}$$

The explicit matrices may be listed in an appendix if desired; in the main text, what matters most is that there are eight linearly independent generators. A useful way to remember this is that Hermitian 3×3 matrices contain nine real parameters, and the traceless condition removes one of them. The Lie algebra of $\mathrm{SU}(3)$ therefore has dimension $8 = 3^2 - 1$.

4.3 Commutation relations and structure constants

The algebra of $\mathrm{SU}(3)$ is defined by

$$[T^a, T^b] = i f^{abc} T^c, \tag{4.4}$$

where f^{abc} are the real antisymmetric structure constants. This is the algebraic heart of the non-Abelian nature of QCD: the generators do not commute.

4.4 Fundamental and adjoint representations

The fundamental representation is the three-dimensional representation in which quark fields transform:

$$q \to Uq, \qquad U \in \mathrm{SU}(3)_c. \tag{4.5}$$

The adjoint representation is the representation carried by the gauge bosons. Its dimension is equal to the number of generators of the group. Since $\mathrm{SU}(3)$ has eight generators, its adjoint representation has dimension eight.

Definition 4.1: Adjoint representation

For a Lie group with generators T^a, the adjoint representation is the representation in which the group acts on its own Lie algebra. In QCD the gluon field is Lie-algebra valued, so the gluons naturally belong to the adjoint representation of $\mathrm{SU}(3)_c$.

4.5 Why the gluon sector has eight gauge bosons

Because the gauge field is Lie-algebra valued,

$$G_\mu(x) = G^a_\mu(x) T^a, \tag{4.6}$$

there is one independent gauge field component G^a_μ for each generator T^a. Since $a = 1, \ldots, 8$, QCD contains eight gluon fields.

A useful pedagogical picture starts from the nine naive color–anticolor combinations

$$|r\bar{r}\rangle,\ |r\bar{g}\rangle,\ |r\bar{b}\rangle,\ |g\bar{r}\rangle,\ |g\bar{g}\rangle,\ |g\bar{b}\rangle,\ |b\bar{r}\rangle,\ |b\bar{g}\rangle,\ |b\bar{b}\rangle.$$

These decompose as

$$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}. \tag{4.7}$$

The singlet combination

$$\frac{1}{\sqrt{3}}\left(|r\bar{r}\rangle + |g\bar{g}\rangle + |b\bar{b}\rangle\right) \tag{4.8}$$

is *not* an $\mathrm{SU}(3)_c$ gluon state. The physical gluons span only the traceless octet directions of the Lie algebra.

$r\bar{g}$ $r\bar{b}$ $g\bar{r}$ $g\bar{b}$

$b\bar{r}$ $b\bar{g}$ $\frac{1}{\sqrt{2}}(r\bar{r} - g\bar{g})$ $\frac{1}{\sqrt{6}}(r\bar{r} + g\bar{g} - 2b\bar{b})$

Eight octet combinations

$\frac{1}{\sqrt{3}}(r\bar{r} + g\bar{g} + b\bar{b})$

color singlet

The singlet is not an $\mathrm{SU}(3)_c$ gluon state

Figure 4: Schematic organization of the eight gluon states of $\mathrm{SU}(3)_c$ as a color octet, contrasted with the absent color-singlet combination.

Remark 4.1: Why the singlet is absent

The color-singlet combination belongs to the identity direction of $U(3)$, not to the traceless algebra of $\mathrm{SU}(3)_c$. Put differently: QCD is built from the generators of $\mathrm{SU}(3)$, and there are eight of them. This representation-theory statement is the precise reason why the formal theory contains eight gluons rather than nine.

Guided checks

1. Verify that the dimension of the Lie algebra of $\mathrm{SU}(3)$ is $3^2 - 1 = 8$.
2. Explain why the naive color–anticolor singlet belongs naturally to $U(3)$ language but not to the traceless $\mathrm{SU}(3)$ gluon sector.
3. State in one sentence the difference between the fundamental representation and the adjoint representation in QCD.

Why the octet statement matters physically: The statement that QCD has eight gluons is sometimes presented as if it were only a counting fact. But it is more than that. The gauge field is not an arbitrary collection of colored vector bosons; it is a field valued in the Lie algebra of $\mathrm{SU}(3)_c$. Because that Lie algebra is eight-dimensional, the gluon sector spans exactly eight independent traceless directions. The missing singlet combination is therefore not absent by phenomenological accident. It is excluded already by the choice of gauge group: QCD is based on $\mathrm{SU}(3)$, not on $\mathrm{U}(3)$.

5 Quark and gluon fields in QCD

5.1 Quark fields with flavor and color labels

For each flavor f, the quark field is a Dirac spinor carrying a color index:

$$q^i_f(x), \qquad i = 1, 2, 3. \tag{5.1}$$

Thus QCD contains both spinor structure and color structure. The spinor structure is inherited from the relativistic fermion theory developed earlier in the course; the color structure is the new internal degree of freedom associated with $\mathrm{SU}(3)_c$.

5.2 Gluon fields as gauge fields of $\mathrm{SU}(3)_c$

The gluon field is introduced exactly as required by local non-Abelian gauge symmetry. It is a Lie-algebra-valued vector field:

$$G_\mu(x) = G^a_\mu(x)T^a. \tag{5.2}$$

The index μ is the Lorentz vector index; the index a labels the eight directions in the Lie algebra. Thus the gluon field is simultaneously a spacetime vector field and an internal color object.

5.3 Gluons in the adjoint representation

Unlike quarks, which carry color in the fundamental representation, the gluons are associated with the adjoint representation. In practical terms, this means that the gauge bosons themselves participate in the non-Abelian structure of the theory rather than merely mediating it passively.

5.4 Physical interpretation of "gluons carry color"

Saying that gluons carry color does not mean that a gluon is simply a quark-like colored particle. Rather, it means that the gauge boson is itself part of the non-Abelian charge structure of the theory. This has a profound consequence: gluons can interact with one another.

There are two complementary languages here. In heuristic color-flow language, a gluon may be pictured as carrying a color and an anti-color index. In the precise representation-theory language, the physical gluons form the adjoint octet of $\mathrm{SU}(3)_c$. The first language is often useful for intuition and Feynman diagrams; the second is the exact group-theoretic statement.

Remark 5.1: "Gluons carry color" — heuristic picture versus exact statement

The phrase "gluons carry color" is shorthand. The exact statement is that the gluon field takes values in the adjoint representation of the $SU(3)_c$ algebra. The popular color–anticolor picture is helpful, but it must be interpreted as a basis-dependent way of visualizing the same octet structure, not as a separate physical principle.

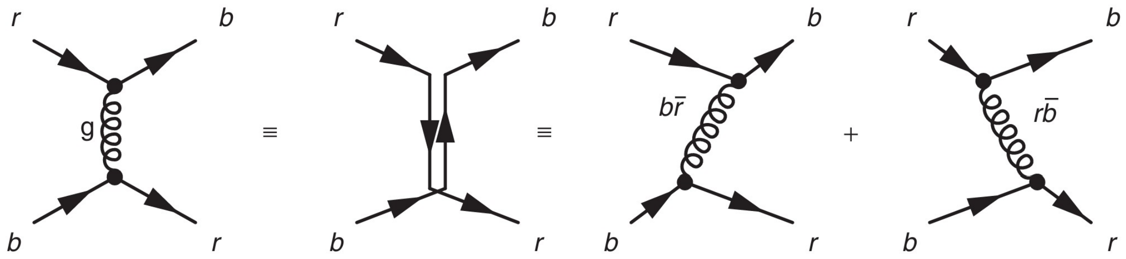


Figure 5: Color flow in quark–quark scattering, useful for illustrating how gluon exchange redistributes color between quark lines.

A useful caution on pictures: The color–anticolor picture is often extremely helpful when drawing color flow in scattering processes, but it should not be confused with the primary formal definition of the gluon field. The exact statement is representation-theoretic: the physical gluons span the adjoint octet of $SU(3)_c$. The color-flow language is a convenient way of visualizing one basis of that octet.

6 The QCD covariant derivative and field-strength tensor

6.1 The QCD covariant derivative

For a quark field transforming in the fundamental representation of $SU(3)_c$, the covariant derivative is

$$D_\mu = \partial_\mu - i g_s T^a G^a_\mu. \tag{6.1}$$

Equivalently, in matrix notation,

$$D_\mu = \partial_\mu - i g_s G_\mu, \qquad G_\mu = G^a_\mu T^a. \tag{6.2}$$

This is the direct QCD specialization of the Yang–Mills covariant derivative already developed in Module 3.

6.2 Gauge covariance of quark fields

Under a local color transformation

$$q(x) \to q'(x) = U(x) q(x), \tag{6.3}$$

the covariant derivative must transform in the same way:

$$D_\mu q(x) \to D'_\mu q'(x) = U(x) D_\mu q(x). \tag{6.4}$$

This condition is much more restrictive than it may first appear. It means that the gauge field cannot transform as an ordinary matter field, because the derivative acting on the spacetime-dependent matrix $U(x)$ produces an extra term. Demanding that this extra term be cancelled fixes the transformation of the matrix-valued gluon field to be

$$G_\mu \to G'_\mu = U G_\mu U^{-1} + \frac{i}{g_s}(\partial_\mu U)U^{-1}, \tag{6.5}$$

or, equivalently,

$$D'_\mu = U D_\mu U^{-1}. \tag{6.6}$$

This is the precise mathematical sense in which the gluon field is forced on us by local color symmetry. One does not first invent a color force and only later decide to write it covariantly. Rather, once $\mathrm{SU}(3)_c$ is taken to be a local symmetry, a connection field with the transformation law above is required in order for derivatives of quark fields to transform consistently.

Example 6.1: Why the gluon transformation law is forced

Starting from $q' = Uq$, one has

$$\partial_\mu q' = (\partial_\mu U) q + U \partial_\mu q.$$

If one now demands $D'_\mu q' = U D_\mu q$, then the unwanted term $(\partial_\mu U) q$ must be cancelled by the transformation of the gauge field. That requirement fixes the inhomogeneous term

$$\frac{i}{g_s}(\partial_\mu U)U^{-1}$$

in G'_μ. The gauge-field transformation law is therefore not guessed; it is enforced by local covariance.

Infinitesimal component form: For an infinitesimal local transformation,

$$U(x) \simeq 1 + i\alpha^a(x) T^a,$$

the matrix transformation law in Eq. (6.5) may be written in component form as

$$G^{a\,\prime}_\mu = G^a_\mu - \frac{1}{g_s}\partial_\mu \alpha^a - f^{abc}\alpha^b G^c_\mu. \tag{6.7}$$

This form is useful because it makes explicit both the inhomogeneous derivative term and the non-Abelian mixing term proportional to the structure constants. The latter has no Abelian analogue and reflects the fact that the gluon field takes values in the Lie algebra of $\mathrm{SU}(3)_c$.

Conceptual pause: the gauge field is a connection, not an added force by hand: This derivation deserves a short pause. In elementary presentations of interactions one sometimes speaks as if a force field is first introduced and only later written in a covariant form. The gauge-theory viewpoint reverses that logic. One begins by demanding local symmetry, and the need for a connection field follows from the requirement that derivatives of matter fields transform consistently. The gluon field is therefore not an optional embellishment of quark theory. It is the mathematically necessary object that allows local color symmetry to be implemented at all.

6.3 Field strength from the commutator $[D_\mu, D_\nu]$

The non-Abelian field-strength tensor is defined through the commutator of covariant derivatives:

$$[D_\mu, D_\nu] = -ig_s G_{\mu\nu}, \tag{6.8}$$

where

$$G_{\mu\nu} = G^a_{\mu\nu} T^a. \tag{6.9}$$

Let us derive this explicitly.

Starting from

$$D_\mu = \partial_\mu - ig_s G_\mu, \qquad D_\nu = \partial_\nu - ig_s G_\nu, \tag{6.10}$$

we compute

$$\begin{aligned}[D_\mu, D_\nu] &= [\partial_\mu - ig_s G_\mu, \partial_\nu - ig_s G_\nu] \\ &= [\partial_\mu, \partial_\nu] - ig_s[\partial_\mu, G_\nu] + ig_s[\partial_\nu, G_\mu] - g_s^2[G_\mu, G_\nu].\end{aligned} \tag{6.11}$$

Since ordinary derivatives commute,

$$[\partial_\mu, \partial_\nu] = 0. \tag{6.12}$$

Acting on a quark field ψ, the remaining terms give

$$[D_\mu, D_\nu]\psi = -ig_s(\partial_\mu G_\nu - \partial_\nu G_\mu)\psi - g_s^2[G_\mu, G_\nu]\psi. \tag{6.13}$$

Therefore,

$$[D_\mu, D_\nu] = -ig_s\left(\partial_\mu G_\nu - \partial_\nu G_\mu - ig_s[G_\mu, G_\nu]\right). \tag{6.14}$$

Hence the matrix-valued field strength is

$$G_{\mu\nu} = \partial_\mu G_\nu - \partial_\nu G_\mu - ig_s[G_\mu, G_\nu]. \tag{6.15}$$

Now substitute

$$G_\mu = G^a_\mu T^a. \tag{6.16}$$

Then

$$[G_\mu, G_\nu] = G^a_\mu G^b_\nu [T^a, T^b] = if^{abc} G^a_\mu G^b_\nu T^c. \tag{6.17}$$

Therefore,

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s f^{abc} G^b_\mu G^c_\nu. \tag{6.18}$$

Covariance of the field strength: Because the field strength is defined through the commutator of covariant derivatives, it transforms covariantly under a local color transformation:

$$G_{\mu\nu} \longrightarrow G'_{\mu\nu} = U G_{\mu\nu} U^{-1}. \tag{6.19}$$

This is the non-Abelian analogue of the good transformation behavior of the electromagnetic field strength. The important difference is that in Yang–Mills theory the field strength is Lie-algebra valued and therefore transforms covariantly rather than remaining strictly invariant component by component.

6.4 Why the non-Abelian term matters physically

The final term in Eq. (6.18),

$$g_s f^{abc} G^b_\mu G^c_\nu, \tag{6.20}$$

is the defining non-Abelian term. It is absent in QED because the Abelian Lie algebra has vanishing commutator. In QCD this term means that the gauge field contributes to its own field strength. That is the formal origin of gluon self-interactions.

It is useful to state the consequence in very plain language. In an Abelian theory, the gauge field is a neutral mediator: it transmits the charge carried by matter, but it does not itself belong to a nontrivial charge multiplet. In QCD the situation is different. Because the gluon field is tied to the adjoint color algebra, the gluon participates in the same color structure that it mediates. This immediately makes the classical field equations nonlinear and, at the quantum level, it changes the vacuum structure of the theory in a way that eventually leads to anti-screening, asymptotic freedom, and the confinement picture.

Take-home message

The commutator $[D_\mu, D_\nu]$ is not algebraic decoration. It is the compact formula from which the nonlinear field strength, the gluon self-interaction vertices, and ultimately the characteristic qualitative behavior of QCD all emerge.

Guided checks

1. Starting from $D_\mu = \partial_\mu - ig_s G_\mu$, reproduce the derivation of Eq. (6.18).
2. Identify explicitly the step in the derivation that would vanish in an Abelian theory.
3. Explain in words why the matrix-valued nature of G_μ makes the field equations nonlinear already at the classical level.

7 The QCD Lagrangian

7.1 Full form of the QCD Lagrangian

The QCD Lagrangian is

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \sum_f \bar{q}_f \left(i\gamma^\mu D_\mu - m_f\right) q_f. \tag{7.1}$$

This compact expression contains the entire classical field content of QCD.

Matrix form versus component form: It is sometimes useful to write the pure-gluon part in matrix notation as

$$\mathcal{L}_{\text{YM}} = -\frac{1}{2} \text{Tr}\left(G_{\mu\nu} G^{\mu\nu}\right). \tag{7.2}$$

Using

$$G_{\mu\nu} = G^a_{\mu\nu} T^a, \qquad \mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab},$$

this becomes

$$-\frac{1}{2}\mathrm{Tr}(G_{\mu\nu}G^{\mu\nu}) = -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu}.$$

This is the same pure-gluon term written in two different notations; the component form used in the main text follows directly from the trace normalization of the generators.

7.2 Gluon kinetic term

The first term,

$$-\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu}, \tag{7.3}$$

is the gauge-field kinetic term. In form it resembles the Maxwell term of QED, but because the field strength itself is nonlinear in the gluon field, this term contains much richer interaction content than the Abelian case.

7.3 Quark kinetic term and flavor sum

The second term contains a sum over quark flavors:

$$\sum_f \bar{q}_f \left(i\gamma^\mu D_\mu - m_f\right) q_f. \tag{7.4}$$

Each quark flavor is described as a Dirac field transforming as a color triplet. The covariant derivative ensures that the kinetic term is locally gauge invariant under $\mathrm{SU}(3)_c$.

7.4 Mass terms and their interpretation

The quark masses appear explicitly as

$$-\sum_f m_f \bar{q}_f q_f. \tag{7.5}$$

These are the quark mass parameters that enter QCD as part of the Standard Model. At the level of the present module, we need only note that the mass term is diagonal in color and respects the color symmetry. The mass term breaks neither local $\mathrm{SU}(3)_c$ invariance nor Lorentz invariance; it merely distinguishes the different quark flavors through their numerical masses. By contrast, a naive gluon mass term of the form $m_G^2 G^a_\mu G^{a\mu}$ would spoil gauge invariance. This is one reason why the gauge bosons of the unbroken color theory remain massless in the Standard Model.

7.5 Interaction content hidden inside the covariant derivative

The interaction between quarks and gluons is already encoded in the covariant derivative. Expanding the fermion term gives

$$\begin{aligned} \bar{q}_f i\gamma^\mu D_\mu q_f &= \bar{q}_f i\gamma^\mu \left(\partial_\mu - ig_s T^a G^a_\mu\right) q_f \\ &= \bar{q}_f i\gamma^\mu \partial_\mu q_f + g_s\, \bar{q}_f \gamma^\mu T^a G^a_\mu q_f. \end{aligned} \tag{7.6}$$

Thus the quark–gluon interaction term is

$$\mathcal{L}_{q\bar{q}g} = g_s\, \bar{q}_f \gamma^\mu T^a G^a_\mu q_f. \tag{7.7}$$

This is worth pausing over. The QCD interaction vertex is not added later as a separate rule; it is already present inside the compact gauge-invariant kinetic term. Once the quark representation and the covariant derivative are fixed, the color structure T^a and the universal coupling g_s follow automatically.

Where the elementary interactions sit: Although the Lagrangian is compact, it is useful to unpack where the interactions actually reside. The fermionic term

$$\sum_f \bar{q}_f\, i\gamma^\mu D_\mu q_f$$

contains both the free quark kinetic term and the quark–gluon interaction obtained by substituting

$$D_\mu = \partial_\mu - ig_s T^a G^a_\mu.$$

Meanwhile, the pure-gluon term

$$-\frac{1}{4}\, G^a_{\mu\nu} G^{a\mu\nu}$$

contains not only the free propagation of the gluon field but also cubic and quartic powers of the gluon field because the non-Abelian field strength is already nonlinear in G^a_μ. Thus the entire elementary interaction structure of QCD is encoded in just two compact gauge-invariant pieces: one matter term and one gauge-field term.

7.6 Why the QCD Lagrangian is simple but highly constraining

Equation (7.1) is compact, but it is not loose. Once the gauge group, field content, and representation assignments are fixed, the allowed structures are tightly restricted. In this sense, QCD is not built by guessing many different strong-interaction terms and later discarding them. Rather, its basic interaction pattern follows from local $\mathrm{SU}(3)_c$ gauge invariance.

Remark 7.1: Gauge invariance fixes more than one might first expect

The QCD Lagrangian is short because the symmetry is restrictive, not because the theory is poor. Once the group, the representations, and locality are chosen, the interaction pattern is highly constrained. That is precisely the lesson already learned in Module 3, now realized in the concrete strong-interaction sector.

7.7 Color current and conserved-current structure

The interaction term of QCD shows immediately which matter current the gluon field couples to. From

$$\mathcal{L}_{\bar{q}qg} = g_s \sum_f \bar{q}_f \gamma^\mu T^a G^a_\mu q_f, \tag{7.8}$$

one reads off the quark color current

$$J^{a\mu}_q \equiv \sum_f \bar{q}_f \gamma^\mu T^a q_f. \tag{7.9}$$

This is the non-Abelian analogue of the electromagnetic current in QED. The important difference is that in QCD the gauge bosons themselves carry color. For that reason, the current structure is more subtle than in the Abelian case.

At the level of the gauge-field equation of motion, the gluon field couples to the quark color current through

$$(D_\nu G^{\nu\mu})^a = g_s J^{a\mu}_q, \tag{7.10}$$

where, for any adjoint-valued object X^a,

$$(D_\nu X)^a \equiv \partial_\nu X^a + g_s f^{abc} G^b_\nu X^c. \tag{7.11}$$

Written out in ordinary derivatives, Eq. (7.10) becomes

$$\partial_\nu G^{a\,\nu\mu} = g_s \left(J^{a\mu}_q - f^{abc} G^b_\nu G^{c\,\nu\mu} \right). \tag{7.12}$$

This motivates the definition of a total color current,

$$J^{a\mu}_{\text{tot}} \equiv J^{a\mu}_q - f^{abc} G^b_\nu G^{c\,\nu\mu}, \tag{7.13}$$

so that the gauge-field equation may be written in the Maxwell-like form

$$\partial_\nu G^{a\,\nu\mu} = g_s J^{a\mu}_{\text{tot}}. \tag{7.14}$$

At this point one sees clearly why QCD differs from QED. In QED the gauge boson does not itself carry the gauge charge, so the matter current is the whole source. In QCD the quark current is only part of the story: the nonlinear gauge-field dynamics supply an additional gluonic contribution to the full color current.

Because $G^{a\,\nu\mu}$ is antisymmetric in ν and μ, Eq. (7.14) immediately implies

$$\partial_\mu J^{a\mu}_{\text{tot}} = 0. \tag{7.15}$$

Thus the full color current is ordinarily conserved, while the source current written in the Yang–Mills form of Eq. (7.10) is naturally expressed through a covariant conservation law rather than an ordinary one.

7.8 Classical equations of motion

The QCD Lagrangian determines the classical field equations for both the quark fields and the gluon field. These equations are worth stating explicitly because they show how the compact Yang–Mills structure becomes a coupled matter–gauge system.

Varying the Lagrangian with respect to $\bar{q}_f$ gives the Dirac equation in a non-Abelian gauge background:

$$\left(i\gamma^\mu D_\mu - m_f\right) q_f = 0. \tag{7.16}$$

Varying with respect to q_f gives the adjoint equation

$$\bar{q}_f \left(i\overleftarrow{D}_\mu \gamma^\mu + m_f\right) = 0, \tag{7.17}$$

where the left-acting covariant derivative is defined by

$$\bar{q}_f \overleftarrow{D}_\mu \equiv \partial_\mu \bar{q}_f + i g_s \bar{q}_f T^a G^a_\mu. \tag{7.18}$$

For the gluon field, one varies with respect to G^a_ν. A convenient intermediate step is the variation of the field strength:

$$\delta G^a_{\mu\nu} = \left(D_\mu \delta G_\nu\right)^a - \left(D_\nu \delta G_\mu\right)^a. \tag{7.19}$$

Using this, the variation of the pure-gluon term becomes, up to a total derivative,

$$\delta\left(-\frac{1}{4} G^a_{\mu\nu} G^{a\,\mu\nu}\right) = \left(D_\mu G^{\mu\nu}\right)^a \delta G^a_\nu. \tag{7.20}$$

Meanwhile, the fermion term contributes

$$\delta \mathcal{L}_{\text{matter}} = g_s \sum_f \bar{q}_f \gamma^\nu T^a q_f \, \delta G^a_\nu. \tag{7.21}$$

Requiring the total variation to vanish therefore gives the Yang–Mills equation of motion,

$$\left(D_\mu G^{\mu\nu}\right)^a = g_s \sum_f \bar{q}_f \gamma^\nu T^a q_f. \tag{7.22}$$

Equation (7.16) shows that the quarks propagate in the background of the color gauge field, while Eq. (7.22) shows that the gluon field is sourced by the color current of the quarks. Together, these equations make explicit the mutual coupling already encoded compactly in the QCD Lagrangian.

8 QCD interaction vertices and gluon self-interactions

8.1 The quark–gluon vertex

As just seen, the fermion kinetic term with covariant derivative produces the interaction

$$g_s \, \bar{q} \gamma^\mu T^a G^a_\mu q. \tag{8.1}$$

This is the quark–gluon vertex. It expresses the statement that gluons couple to color charge.

8.2 The triple-gluon and quartic-gluon structures

The non-Abelian field-strength tensor

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s f^{abc} G^b_\mu G^c_\nu \tag{8.2}$$

contains both a term linear in the gluon field and a term quadratic in the gluon field. When substituted into the gauge kinetic term

$$-\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu}, \tag{8.3}$$

it produces not only the free kinetic part, but also cubic and quartic powers of the gluon field. Schematically,

$$G^a_{\mu\nu} G^{a\mu\nu} \sim (\partial G)^2 + g_s(\partial G)GG + g_s^2 GGGG. \tag{8.4}$$

The second term generates the triple-gluon vertex, while the last generates the quartic-gluon vertex.

Example 8.1: How the nonlinear term generates gluon self-interactions

Write the field strength as a sum of a linear and a nonlinear piece,

$$G^a_{\mu\nu} = (G^a_{\mu\nu})_{\text{lin}} + (G^a_{\mu\nu})_{\text{nl}} \equiv (\partial_\mu G^a_\nu - \partial_\nu G^a_\mu) + g_s f^{abc} G^b_\mu G^c_\nu.$$

Then the gauge kinetic term contains

$$G^a_{\mu\nu} G^{a\mu\nu} = (G^a_{\mu\nu})_{\text{lin}} (G^{a\mu\nu})_{\text{lin}} + 2(G^a_{\mu\nu})_{\text{lin}} (G^{a\mu\nu})_{\text{nl}} + (G^a_{\mu\nu})_{\text{nl}} (G^{a\mu\nu})_{\text{nl}}.$$

The middle term is cubic in the gluon field and gives the ggg interaction. The last term is quartic and gives the $gggg$ interaction. This is the cleanest structural way to see that gluon self-interactions are built directly into the Yang–Mills Lagrangian.

8.3 Why this does not happen in QED

In QED the field-strength tensor is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{8.5}$$

There is no term quadratic in the photon field because the Abelian algebra is trivial. Consequently, $F_{\mu\nu}F^{\mu\nu}$ contains no cubic or quartic gauge-field interactions. Photons do not self-interact at tree level. In QCD, gluons do.

8.4 Structural summary of the basic QCD vertices

The figure below provides a compact summary of the basic tree-level interaction structures contained in the QCD Lagrangian. The first vertex is the quark–gluon coupling, which describes the interaction of colored matter fields with the gluon gauge field. The second and third vertices are qualitatively new relative to QED: they show that gluons can interact directly with one another through triple-gluon and quartic-gluon couplings. This is the practical diagrammatic expression of the non-Abelian $SU(3)_c$ gauge structure and is the reason why QCD is not merely "QED with a different charge," but a theory with genuinely different dynamical behavior.

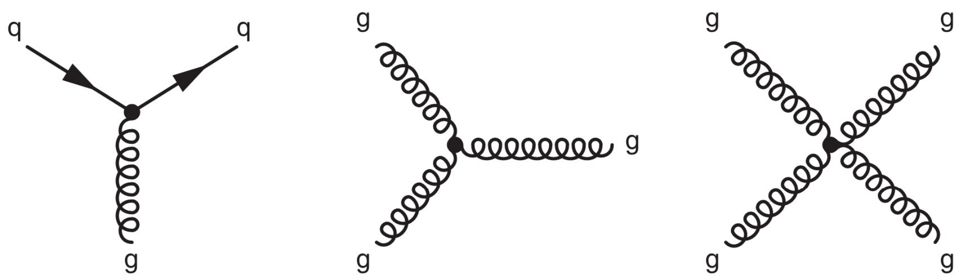


Figure 6: The three basic tree-level interaction structures of QCD: the quark–gluon vertex, the triple-gluon vertex, and the quartic-gluon vertex. The last two are direct consequences of the non-Abelian gauge structure and have no tree-level analogue in QED.

This is why the gluon self-interaction terms are not a decorative formal detail: they are the distinctive structural ingredient from which the characteristic physics of QCD follows.

8.5 Why gluon self-interactions matter

At this formal stage, the most important lesson is simply this: the strong interaction is not mediated by a neutral spectator field. Because the gauge bosons themselves participate in the non-Abelian structure, QCD possesses a dynamical richness that QED does not. Later we will see that this formal fact is the starting point for the running of α_s, asymptotic freedom, and the qualitative picture of confinement.

An explicit unpacking of $\mathcal{L}_{\text{QCD}}$. As a compact summary of the preceding discussion, it is useful to display the QCD Lagrangian not only in its compact gauge-invariant form, but also in a more explicit expanded form that makes visible where the free terms and the elementary interaction terms actually sit. To do this, define the linear part of the gluon field strength by

$$G^{a,\text{lin}}_{\mu\nu} \equiv \partial_\mu G^a_\nu - \partial_\nu G^a_\mu. \tag{8.6}$$

Then the QCD Lagrangian may be unpacked as

$$\begin{aligned}\mathcal{L}_{\text{QCD}} = \sum_f \bar{q}_f \left(i\gamma^\mu \partial_\mu - m_f\right) q_f + g_s \sum_f \bar{q}_f \gamma^\mu T^a G^a_\mu q_f - \frac{1}{4} G^{a,\text{lin}}_{\mu\nu} G^{a,\text{lin}\,\mu\nu} \\ - \frac{1}{2} g_s f^{abc} G^{a,\text{lin}}_{\mu\nu} G^{b\mu} G^{c\nu} - \frac{1}{4} g_s^2 f^{abc} f^{ade} G^b_\mu G^c_\nu G^{d\mu} G^{e\nu}.\end{aligned} \tag{8.7}$$

The first term is the free quark kinetic term, the second is the quark–gluon interaction, the third is the free gluon kinetic term, and the last two terms generate the triple-gluon and quartic-gluon self-interactions. Equation (8.7) should be read as an explicit unpacking of Eq. (7.1), not as a replacement for the compact gauge-invariant expression.

Guided checks

1. Starting from Eq. (7.7), identify the color matrix that appears in the quark–gluon vertex.
2. Explain why the ggg and $gggg$ vertices originate from the gauge-field kinetic term rather than from the fermion term.
3. State in one sentence why there is no direct QED analogue of the ggg vertex.

Bridge forward: At this point the student should notice that the difference between QED and QCD is no longer merely a difference in vocabulary or in the number of gauge bosons. Once the gauge group is non-Abelian, the covariant derivative becomes matrix-valued, the field strength becomes nonlinear, and self-interaction vertices of the gauge bosons become unavoidable. Before turning to the direct QED–QCD comparison, however, it is useful to insert one short orientation remark on gauge fixing, since perturbative calculations with the gluon field require a careful treatment of gauge redundancy.

9 A brief comment on gauge fixing and perturbative QCD

The classical gauge-field description contains redundant variables: gauge-related field configurations represent the same physical state. In perturbative calculations one therefore supplements the gauge-invariant Lagrangian by a gauge-fixing term so that propagators are well defined and Feynman-diagram expansions can be carried out consistently. At an introductory level it is enough to remember two points. First, gauge fixing is a calculational necessity rather than a change in the physical content of the theory. Second, in non-Abelian gauge theory this procedure is subtler than in QED and leads, in covariant quantization, to the appearance of ghost fields in intermediate perturbative expressions. We do not develop the full BRST or Faddeev–Popov formalism here, but students should recognize that the perturbative treatment of gluons requires more care than the corresponding photon story.

As a concrete orientation example, in a covariant gauge one often adds

$$\mathcal{L}_{\text{gf}} = -\frac{1}{2\xi}(\partial_\mu G^{a\mu})^2, \tag{9.1}$$

where ξ is a gauge parameter. Physical observables are gauge independent even though intermediate expressions may depend on the chosen gauge. The practical message is simple: the gauge potential is a redundant description of the physical gluon degrees of freedom, and this redundancy has to be handled carefully before perturbation theory can be used consistently.

In a full covariantly quantized non-Abelian perturbative treatment, this gauge-fixing term is accompanied by the corresponding Faddeev–Popov ghost term; it is omitted here only because the present subsection is meant as an orientation remark rather than a full quantization formalism.

Remark 9.1: Gauge fixing changes the description, not the physics

Gauge fixing removes redundancy from the field description; it does not change the physical content of the theory. This is why gauge-dependent intermediate expressions can still lead to gauge-invariant observable results.

10 Abelian versus non-Abelian gauge theory: QED compared with QCD

10.1 The two field-strength tensors side by side

The contrast between QED and QCD may be summarized by comparing the two field-strength tensors:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{10.1}$$

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s f^{abc} G^b_\mu G^c_\nu. \tag{10.2}$$

The second contains a genuinely new structure: the gauge field appears inside its own field strength.

10.2 Gauge bosons and gauge charge

In QED, the photon does not carry electric charge. In QCD, the gluon field transforms in the adjoint structure associated with the color algebra, and this is the formal reason why gluons can self-interact while photons do not.

10.3 Linear versus nonlinear gauge dynamics

The equations of motion of Abelian gauge theory are linear in the gauge field at the classical level. Non-Abelian gauge theory is nonlinear already before quantization because the gauge-field sector contains products of the gauge fields themselves.

10.4 Conceptual summary

Feature	QED	QCD
Gauge group	U(1)	$SU(3)_c$
Gauge bosons	one photon	eight gluons
Gauge boson carries gauge charge?	no	yes, in the adjoint sense
Field strength	linear in A_μ	nonlinear in G^a_μ
Tree-level gauge-boson self-interactions	absent	present
Typical asymptotic charged states	electrons, muons, protons, . . .	no free colored states observed

The difference between QED and QCD is not merely that one has one gauge boson while the other has eight. The deeper distinction is that QED is Abelian while QCD is non-Abelian. From this follow:

- the matrix-valued covariant derivative,
- the commutator term in the field strength,
- the triple-gluon and quartic-gluon vertices,
- the qualitatively different short-distance and long-distance dynamics of the theory.

Take-home message
The defining formal fingerprint of QCD is the non-Abelian term in the field-strength tensor. Once that term is present, the gauge bosons self-interact, and the strong interaction becomes a theory with dynamics fundamentally different from electromagnetism.

Part II. Physical QCD: From Running Coupling to Collider Phenomenology

11 Running coupling and the scale dependence of QCD

11.1 Why couplings run in quantum field theory

In classical field theory a coupling constant may look like a single fixed parameter. Quantum field theory changes that intuition. The vacuum is not empty in a naive classical sense; it is populated by virtual fluctuations that polarize the vacuum and modify the effective interaction seen by a probe. For this reason, the coupling extracted from experiment depends on the characteristic scale of the process.

In QCD this is not a minor correction. The scale dependence of the strong coupling is one of the defining physical properties of the theory. It is the reason the same QCD Lagrangian can describe both approximately free short-distance partons and strongly bound long-distance hadrons.

At a practical level, this means that the strong interaction cannot be characterized by a single number that applies equally well in all regimes. One instead defines the running coupling

$$\alpha_s(\mu) \equiv \frac{g_s^2(\mu)}{4\pi}, \tag{11.1}$$

where μ denotes the renormalization scale and, in a physical application, is chosen to be of the order of the characteristic hard scale of the process.

Definition 11.1: Running coupling

A running coupling is an effective interaction strength whose numerical value depends on the energy or momentum scale at which a process is probed. In renormalized QCD this scale dependence is encoded in $\alpha_s(\mu)$.

One should not imagine that the "true" coupling is changing in time. Rather, different experiments probe the same theory at different resolutions. The renormalization scale keeps track of that resolution. Large μ corresponds to short-distance structure; small μ corresponds to long-distance structure.

11.2 The QCD beta function at leading order

The scale dependence of the coupling is encoded in the beta function,

$$\beta(\alpha_s) \equiv \mu \frac{\mathrm{d}\alpha_s}{\mathrm{d}\mu}. \tag{11.2}$$

For a general SU(N_c) gauge theory with N_f active quark flavors, the leading-order result is

$$\beta(\alpha_s) = -\frac{\beta_0}{2\pi}\alpha_s^2 + O(\alpha_s^3), \qquad \beta_0 = \frac{11}{3}N_c - \frac{2}{3}N_f. \tag{11.3}$$

For QCD itself, $N_c = 3$, so

$$\beta_0 = 11 - \frac{2}{3}N_f. \tag{11.4}$$

At MSc level the most important point is not a full loop derivation, but the physical meaning of the two contributions to β_0. Quark loops contribute with the same qualitative sign as ordinary screening in QED. The genuinely non-Abelian gluon contribution has the opposite sign. Since gluons themselves carry color and self-interact, the gluon sector dominates and drives the theory toward anti-screening.

Remark 11.1: What is meant by screening and anti-screening?

In QED vacuum polarization tends to reduce the effective charge seen at long distance and increase the measured coupling at short distance: this is screening. In QCD the gluon contribution reverses that intuition. The effective color charge becomes weaker when probed at shorter distance: this is anti-screening.

Tong's discussion of anti-screening and paramagnetism is especially useful here: the sign is not an algebraic accident but a reflection of how the non-Abelian gauge field itself responds to the probe. In this sense, "gluons talk to gluons," and that changes the vacuum qualitatively.

11.3 One-loop running of $\alpha_s(Q^2)$

Integrating the leading-order beta function gives the familiar one-loop running form

$$\alpha_s(Q^2) \simeq \frac{4\pi}{\beta_0 \ln\left(Q^2/\Lambda_{\text{QCD}}^2\right)}. \tag{11.5}$$

Here Q denotes the characteristic momentum-transfer scale and Λ_{QCD} is the dynamically generated QCD scale.

A useful equivalent form is

$$\frac{1}{\alpha_s(Q^2)} \simeq \frac{\beta_0}{4\pi}\ln\left(\frac{Q^2}{\Lambda_{\text{QCD}}^2}\right). \tag{11.6}$$

Written this way, one sees immediately that increasing Q increases the logarithm and therefore decreases α_s.

Example 11.1: A quick sign check

Suppose $N_f = 5$, appropriate for many hard collider processes below the top threshold. Then

$$\beta_0 = 11 - \frac{10}{3} = \frac{23}{3} > 0.$$

Hence the leading beta function is negative, $\beta(\alpha_s) < 0$, so α_s decreases as the scale is raised. That is the operational content of asymptotic freedom.

11.4 Why the sign of the beta function is the key result

The single most important lesson of Eqs. (11.3)–(11.5) is the sign. If the beta function had the QED-like sign, QCD would become stronger at shorter distances and the parton picture would collapse. Instead, the negative sign makes short-distance QCD calculable.

There is a useful conceptual hierarchy here.

- The *formal* origin of the sign is the non-Abelian gauge structure.
- The *physical* origin is that gluon self-interactions make the vacuum anti-screening.
- The *phenomenological* consequence is that hard scattering can be treated perturbatively.

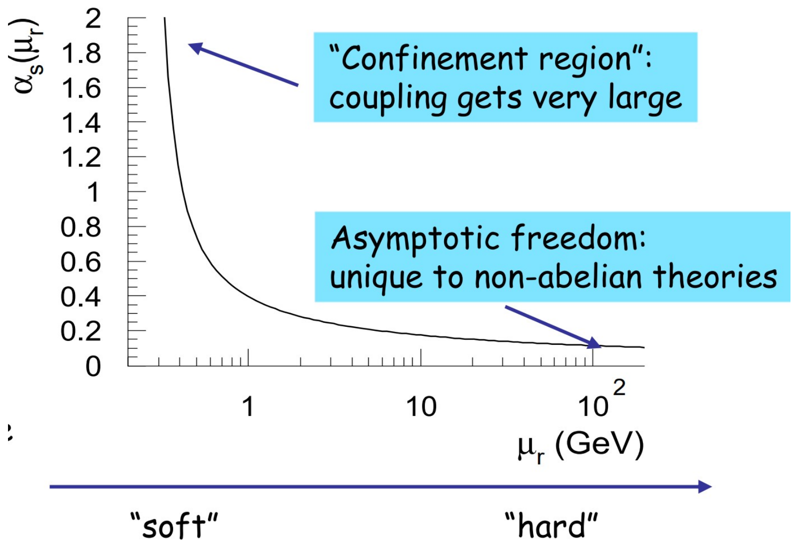


Figure 7: Qualitative running of the strong coupling $\alpha_s(\mu_r)$, showing the transition from the soft, strongly coupled regime to the hard, perturbative regime. At low scales the coupling becomes large and nonperturbative physics dominates, while at high scales the decrease of α_s is the operational signature of asymptotic freedom.

This schematic picture should be read only as a qualitative guide. A more realistic summary, based on measurements of α_s extracted from different physical processes over a wide range of scales, is shown in the next figure.

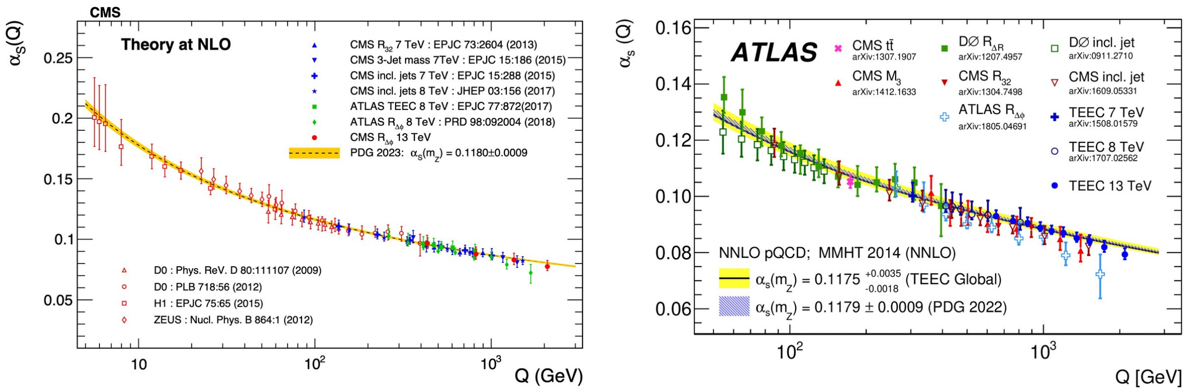


Figure 8: Representative world-summary plot of the running strong coupling α_s extracted from different processes over a wide range of momentum scales. The measured decrease with increasing Q provides direct evidence for asymptotic freedom.

11.5 The meaning of Λ_{QCD}

Equation (11.5) should not be read as saying that perturbation theory gives a literal pole of physical significance at $Q = \Lambda_{\text{QCD}}$. Rather, Λ_{QCD} marks the scale at which the perturbative expansion ceases to be trustworthy and genuinely strong-coupling physics becomes unavoidable.

Rearranging Eq. (11.5) gives the qualitative relation

$$\Lambda_{\text{QCD}} \sim Q \exp\left[-\frac{2\pi}{\beta_0 \alpha_s(Q)}\right]. \tag{11.7}$$

Its conceptual significance is deeper than its numerical use: a classically scale-free Yang–Mills theory acquires a dynamical dimensionful scale after renormalization.

Remark 11.2: Dimensional transmutation

In classical QCD the coupling is dimensionless and no intrinsic mass scale appears in the Yang–Mills term. Quantum mechanically the renormalized theory is more subtle: one trades the dimensionless coupling for a dynamically generated scale, Λ_{QCD}. This phenomenon is called dimensional transmutation. Quevedo's strong-interactions notes are especially useful for presenting this logic in a compact formula-level way.

11.6 What this means for practical calculations

The running of α_s divides QCD into two broad regimes:

- a **hard regime**, where α_s is small enough for perturbation theory to be meaningful,
- a **soft regime**, where the coupling becomes large and hadronic language becomes unavoidable.

Guided checks

1. Why is the sign of the QCD beta function more important pedagogically here than a full one-loop derivation?
2. Which contribution tends to screen color charge, and which contribution tends to anti-screen it?
3. Why should Eq. (11.5) not be interpreted as a literally reliable low-scale formula?

Take-home message

The most important physical consequence of gluon self-interactions is not merely that QCD has extra vertices. It is that the coupling runs in the opposite qualitative direction from QED: QCD becomes weak at short distances and strong at long distances.

12 Asymptotic freedom

12.1 Weak coupling at short distances

Asymptotic freedom is the statement that the strong interaction becomes weak at short distances, or equivalently at large momentum transfer. From Eq. (11.5),

$$Q^2 \to \infty \qquad \Longrightarrow \qquad \alpha_s(Q^2) \to 0. \tag{12.1}$$

This does not mean that the strong interaction disappears. It means that the expansion parameter governing hard-process calculations becomes small enough that perturbative QCD becomes quantitatively useful.

12.2 Physical meaning for hard scattering

The phrase "quasi-free parton" should be understood carefully. A quark inside a proton is not a free asymptotic particle. Rather, during a sufficiently short-duration hard scattering, the probe interacts with a single partonic constituent before long-distance QCD has time to reorganize the final state into hadrons. This is the physical logic behind the parton model.

A useful timescale picture is the following. If the probe transfers momentum Q, then the interaction probes transverse distances of order

$$r_\perp \sim \frac{1}{Q}. \tag{12.2}$$

Large Q therefore means a short-distance interrogation of the hadron.

Example 12.1: Why "quasi-free" does not mean "free"

In deep inelastic scattering the struck quark can behave almost as if it were free during the hard vertex, but it never emerges into the detector as an isolated color charge. The hard subprocess is perturbative; the late-time final state is still hadronic. This is precisely why asymptotic freedom and confinement are not contradictory.

12.3 Why perturbation theory works at high Q^2

Perturbation theory is useful when higher powers of the coupling are genuinely suppressed. In QCD this is not true at all scales, but it is true in a hard process with large Q^2, large invariant mass, or large jet transverse momentum. This is why the language of quarks and gluons becomes predictive in short-distance scattering, even though the asymptotic observed states remain hadronic.

Guided checks

1. Explain in one sentence why asymptotic freedom is a short-distance statement rather than a statement about free colored particles in the detector.
2. Why does a large momentum transfer correspond to a short-distance probe?
3. Why is asymptotic freedom the conceptual bridge from the QCD Lagrangian to the parton model?

Take-home message
Asymptotic freedom is what makes the internal quark–gluon structure of hadrons experimentally accessible. Without it, there would be no clean parton picture of hard scattering.

13 Infrared growth of α_s and the onset of strong coupling

13.1 Growth of the coupling at low scales

The same running formula that leads to asymptotic freedom also tells us what happens when the probing scale is lowered. As Q approaches the hadronic scale, the denominator of Eq. (11.5) becomes smaller and the coupling grows:

$$Q^2 \downarrow \qquad \Longrightarrow \qquad \alpha_s(Q^2) \uparrow . \tag{13.1}$$

Eventually α_s becomes of order one and the perturbative expansion loses its parametric control.

13.2 Breakdown of perturbation theory

When $\alpha_s \ll 1$, a higher-order contribution is typically suppressed by extra powers of the coupling. When $\alpha_s \sim 1$, there is no small parameter that guarantees the dominance of the first few terms. This is the practical meaning of the breakdown of perturbation theory. It is not that the theory ceases to exist; it is that the weak-coupling calculational language ceases to be reliable.

Remark 13.1: Strong coupling is not just "the same theory with bigger numbers"

At strong coupling the qualitative physics can change. Tong emphasizes exactly this point in his discussion of the mass gap: when the coupling is large, the quantum theory need not resemble the classical field picture in any simple way. QCD at low energies is the archetypal example.

13.3 Dimensional transmutation at orientation level

The emergence of Λ_{QCD} means that the quantum theory dynamically generates a hadronic scale. This is why low-energy QCD is governed by characteristic energies of order a few hundred MeV rather than by a dimensionless coupling alone. In that region, the natural language shifts from elementary colored fields to bound states, spectra, and hadronic effective descriptions.

13.4 What can and cannot be computed perturbatively

At this stage it is helpful to separate two broad classes of QCD questions.

Perturbative questions: These involve sufficiently hard scales that quarks and gluons can be treated as the relevant active degrees of freedom. Examples include inclusive jet production at high p_T, hard Drell–Yan production, and the short-distance part of DIS.

Nonperturbative questions: These involve hadron masses, confinement, hadronization, low-energy bound states, and many aspects of the QCD vacuum. Here perturbation theory is inadequate and one relies on lattice QCD, effective theories, symmetry arguments, or phenomenological models.

Guided checks

1. Why does the failure of perturbation theory *not* mean that QCD fails as a theory?
2. Give one example of a perturbative QCD question and one example of a nonperturbative QCD question.
3. Why does the emergence of Λ_{QCD} already prepare the ground for confinement?

Take-home message

The infrared regime of QCD is not merely quantitatively harder than the ultraviolet regime. It is the regime in which hadrons, confinement, and other genuinely nonperturbative phenomena become unavoidable.

14 Confinement and color singlets

14.1 Statement of the confinement phenomenon

Color confinement is the statement that states carrying nonzero net color charge are not observed as freely propagating asymptotic particles. Nature presents us with hadrons and jets, not isolated quarks or isolated gluons.

Definition 14.1: Confinement

Confinement is the physical phenomenon that colored excitations do not appear as freely observable asymptotic states. The observed long-lived strong-interaction spectrum is organized into color-singlet states.

14.2 Why isolated quarks and gluons are not observed

It is important to separate two logically different claims.

- **Physical claim:** confinement is strongly supported by experiment, phenomenology, and lattice calculations.
- **Mathematical claim:** a complete analytic proof of confinement in real-world QCD is not known.

This distinction is important pedagogically because it avoids overstating what has been proved while still being completely honest about the empirical strength of the confinement picture.

Tong's strong-force chapter is especially useful here because it couples the confinement discussion to the mass gap and makes clear that strong-coupling Yang–Mills theory is qualitatively different from classical Maxwell intuition.

Remark 14.1: Confinement is stronger than "many bound states exist"

QED also has bound states such as atoms, yet it still allows isolated electrons. Confinement means something stronger: free color-charged states are absent from the asymptotic spectrum. Quevedo makes this contrast with QED especially clearly in his discussion of color singlets and strong interactions.

14.3 Mesons and baryons as color-singlet states

The simplest and most important color-singlet hadronic combinations emphasized in this module are mesons and baryons. Schematically,

$$\psi_c(q\bar{q}) = \frac{1}{\sqrt{3}}(r\bar{r} + g\bar{g} + b\bar{b}), \tag{14.1}$$

and

$$\psi_c(qqq) = \frac{1}{\sqrt{6}}(rgb - rbg + gbr - grb + brg - bgr) \propto \epsilon_{ijk} q^i q^j q^k. \tag{14.2}$$

These are not merely mnemonic wavefunctions; they are group-theoretic expressions of color neutrality.

Example 14.1: Why color singlets matter experimentally

In a hard proton–proton collision the short-distance subprocess may involve colored partons. Yet the final long-lived particles entering the detector are colorless hadrons. The color-singlet requirement is therefore not just a spectroscopy remark; it is built into the appearance of every QCD event.

14.4 Flux-tube intuition and the growth of the potential

A useful qualitative picture of confinement comes from asking what happens when a quark and antiquark are pulled apart. In QED the electric field spreads through space and the Coulomb potential falls as $-1/r$. In QCD the non-Abelian self-interaction of the gauge field is believed to squeeze the chromoelectric field into a roughly tube-like region. If the energy density per unit length is approximately constant, then the energy grows approximately linearly with the separation,

$$V(r) \sim \kappa r, \tag{14.3}$$

where κ is the string tension.

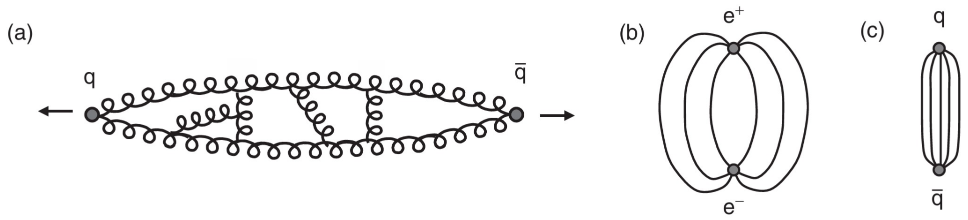


Figure 9: Qualitative picture of the effect of gluon self-interactions on the long-range QCD force, motivating the flux-tube intuition for confinement.

14.5 String breaking and pair creation

If the quark–antiquark separation continues to grow, the energy stored in the flux tube eventually becomes large enough that creating a new $q\bar{q}$ pair from the vacuum is energetically favored. Instead of isolating a single quark, the system reorganizes into two color-singlet subsystems. This is the qualitative string-breaking picture.

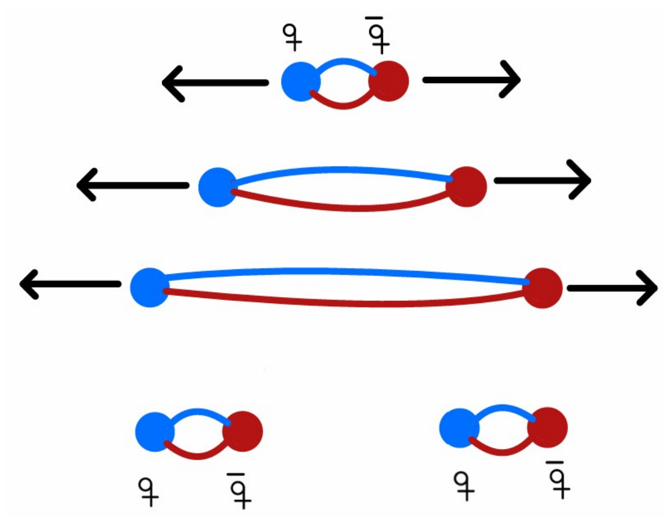


Figure 10: Qualitative picture of string breaking: stretching a quark–antiquark system eventually favors the creation of a new $q\bar{q}$ pair rather than an isolated color charge.

Guided checks

1. Why is confinement a stronger statement than the existence of bound states?
2. Why does the string-breaking picture prevent the isolation of a single quark?
3. State clearly the difference between "confinement is physically established" and "confinement has a complete analytic proof."

Take-home message

Confinement tells us that the colored fields appearing in the QCD Lagrangian do not survive as free asymptotic particles. The physical strong-interaction world is organized into color-singlet hadrons and jets.

15 Hadrons and the emergence of bound states

15.1 From fundamental fields to observed particles

The QCD Lagrangian is written in terms of quark and gluon fields, but the observed particles are hadrons. This is one of the deepest conceptual lessons of strong-coupling physics: the fundamental fields of a theory need not coincide with the asymptotic particles that appear experimentally.

15.2 Mesons

Mesons are color-singlet bound states of a quark and an antiquark. At the level of color algebra this is the statement that

$$3 \otimes \bar{3} = 8 \oplus 1, \tag{15.1}$$

with only the singlet component corresponding to an isolated physical hadron.

15.3 Baryons

Baryons are color-singlet states built from three quarks. Their color wavefunction is totally antisymmetric,

$$\psi_c(qqq) \propto \epsilon_{ijk} q^i q^j q^k. \tag{15.2}$$

This color antisymmetry is essential for the consistency of the full baryon wavefunction once spin, flavor, and spatial structure are included.

15.4 Light hadrons versus heavy quarkonia

It is useful to distinguish between two broad classes of hadronic systems.

- **Light hadrons**, built mainly from u, d, s quarks, are strongly relativistic systems in which confinement-scale dynamics dominate.

- **Heavy quarkonia**, such as $c\bar{c}$ and $b\bar{b}$, often admit a more potential-model-like description because the heavy masses introduce additional scales.

15.5 A qualitative heavy-quark potential

A widely used phenomenological form for the heavy-quark potential is

$$V(r) \approx -\frac{4}{3}\frac{\alpha_s}{r} + \kappa r. \tag{15.3}$$

The first term is Coulomb-like and dominates at short distance, where asymptotic freedom permits a quasi-perturbative interpretation. The second term is linear and represents the confining long-distance part.

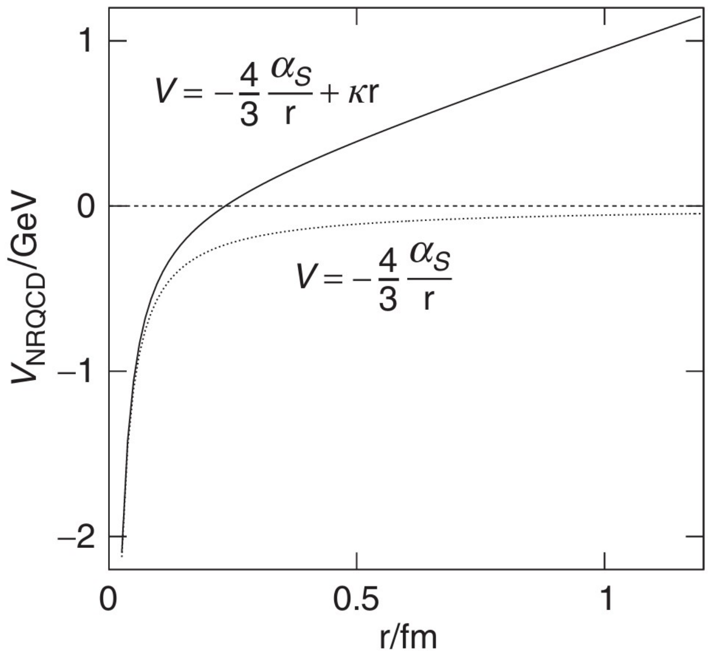


Figure 11: Approximate heavy-quark potential showing the short-distance Coulomb-like term and the long-distance linear confining term.

Remark 15.1: A warning about potential models

Equation (15.3) is a useful phenomenological guide, especially for heavy quarkonia. It should not be mistaken for an exact universal derivation of all hadron physics. For light hadrons, relativistic motion and genuinely nonperturbative dynamics are too important for such a simple picture to be more than qualitative.

15.6 Why hadron masses are not simple sums of quark masses

A naive sum of bare quark masses does not explain the masses of most hadrons. In light hadrons especially, a substantial part of the mass is dynamical and associated with QCD binding, confinement-scale energy, and the structure of the strongly interacting vacuum. For heavy quark systems the explicit quark masses play more visible roles, but even there binding effects are not negligible.

Example 15.1: Why the proton mass is a QCD lesson

The proton contains only light valence quarks, yet its mass is close to a GeV. This is a strong reminder that most visible-matter mass is not explained by adding small current quark masses, but by QCD dynamics.

16 The parton model

16.1 The fast proton as a collection of quasi-free constituents

Part I and the preceding physical sections established two apparently opposite facts about QCD. At low energy, quarks and gluons are not seen as free asymptotic states. At short distance, asymptotic freedom allows a hard probe to interact with a constituent as if it were nearly free during the scattering. The parton model is the language that reconciles these statements in the high-energy regime.

In the infinite-momentum or very highly boosted picture, a fast proton is described during the hard interaction as a collection of quasi-free constituents called partons. In modern QCD language, the partons are quarks, antiquarks, and gluons.

Definition 16.1: Parton

A parton is a constituent degree of freedom of a fast hadron that participates in a hard short-distance scattering. In QCD the relevant partons are quarks, antiquarks, and gluons.

16.2 Momentum fractions and Bjorken x

In a highly boosted picture the momentum of the struck constituent is written approximately as a fraction of the proton momentum,

$$p^{\mu}_{\text{parton}} \approx x\, p^{\mu}_{\text{proton}}, \qquad 0 < x < 1. \tag{16.1}$$

The variable x is then interpreted as the momentum fraction carried by the struck parton.

16.3 Why the parton model works at high Q^2

The parton model works because the hard probe resolves the proton on a timescale short compared with the long-distance strong dynamics that binds the hadron. Asymptotic freedom makes the interaction local and weak enough that the struck constituent can be treated approximately as a free particle during the hard vertex.

16.4 Partons are more than valence quarks

A proton is not simply three static valence quarks each carrying one third of the momentum. QCD radiation, gluon splitting, and sea-quark production mean that the proton contains

- valence quarks,
- sea quarks and antiquarks,
- gluons carrying a substantial share of the proton momentum.

Example 16.1: Why Bjorken x is not "the valence-quark label"

A measurement at some value of Bjorken x does not mean the probe has isolated a static valence quark of that label. The same kinematic variable can be associated with valence-quark, sea-quark, or gluon-dominated regions, depending on the process and scale.

Guided checks

1. Why does the parton model not contradict confinement?
2. Why are gluons counted as partons even though the simplest DIS photon couples directly only to electric charge?
3. Explain in one sentence why the parton model is conceptually tied to asymptotic freedom.

Take-home message

The parton model is not a replacement for QCD. It is the high-energy language in which QCD becomes experimentally visible inside hadrons.

17 Deep inelastic scattering (DIS) as a probe of proton structure

17.1 Electron–proton inelastic scattering

Deep inelastic scattering is historically and conceptually the cleanest window into proton structure. In its simplest neutral-current form,

$$e^- p \to e^- X, \tag{17.1}$$

where X denotes an inclusive hadronic final state rather than a single fixed exclusive channel.

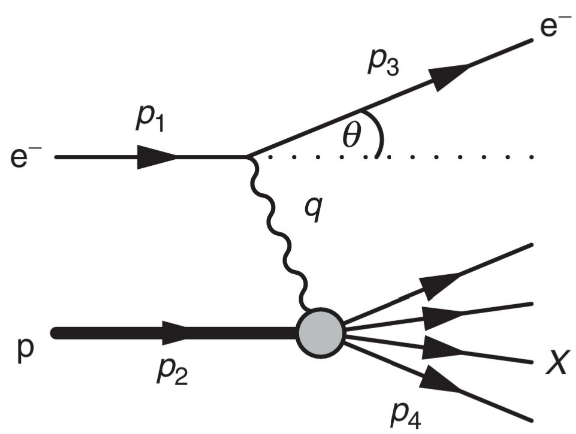


Figure 12: Electron–proton inelastic scattering as the basic deep inelastic scattering process.

17.2 Basic kinematics

Let the incoming and outgoing electron four-momenta be k and k', and let the incoming proton four-momentum be p. The momentum transferred to the hadronic system is

$$q = k - k'. \tag{17.2}$$

Because the exchanged boson is highly virtual,

$$q^2 < 0, \tag{17.3}$$

so one defines the positive variable

$$Q^2 \equiv -q^2. \tag{17.4}$$

The two standard dimensionless DIS variables are

$$x \equiv \frac{Q^2}{2p \cdot q}, \qquad y \equiv \frac{p \cdot q}{p \cdot k}. \tag{17.5}$$

A further useful variable is the invariant mass of the hadronic final state,

$$W^2 \equiv (p + q)^2 = m_p^2 + 2p \cdot q - Q^2 = m_p^2 + Q^2 \left(\frac{1}{x} - 1 \right). \tag{17.6}$$

Large W corresponds to a genuinely inelastic hadronic final state rather than simple elastic scattering.

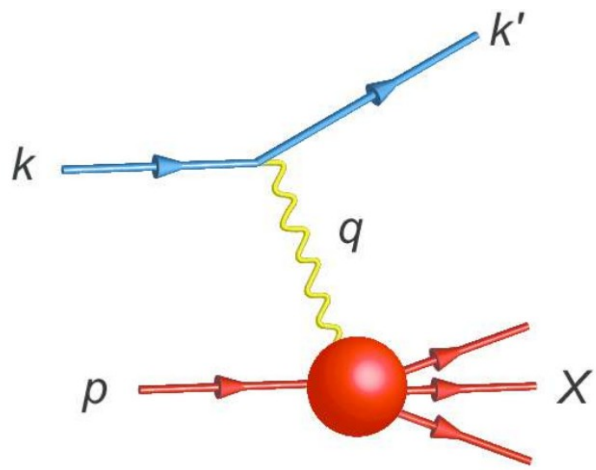


Figure 13: Schematic kinematics of deep inelastic scattering. An incoming lepton with momentum k scatters to k' by exchanging a virtual boson with momentum q with a proton of momentum p, producing an inclusive hadronic final state X.

17.3 Physical meaning of the DIS variables

Each of the DIS variables carries clear physical information.

- Q^2 measures the resolving power of the probe: large Q^2 corresponds to short distance.
- x becomes the momentum fraction of the struck parton in the Bjorken scaling limit.

- y measures the inelasticity: roughly speaking, the fractional energy transfer from the electron to the hadronic system in the proton rest frame.

The relation

$$r \sim \frac{1}{Q} \tag{17.7}$$

encodes the intuitive statement that larger momentum transfer probes smaller transverse structure.

Example 17.1: A worked interpretation of large Q^2 and moderate x

If a DIS event has large Q^2, the probe has short-distance resolving power. If, in addition, the event lies at moderate x, the struck constituent is often interpreted as a valence-dominated parton carrying an appreciable fraction of the proton momentum. At small x, by contrast, sea quarks and gluons become increasingly important.

17.4 Structure functions and the parton picture

At hadron level, unpolarized DIS is parameterized by structure functions such as $F_1(x, Q^2)$ and $F_2(x, Q^2)$. Their conceptual role is to encode the internal structure of the proton as seen by the virtual photon. In the simplest quark–parton model,

$$F_2(x, Q^2) = x \sum_q e_q^2 \left[q(x, Q^2) + \bar{q}(x, Q^2)\right], \tag{17.8}$$

and, for scattering from spin-$\frac{1}{2}$ point-like partons, one finds the Callan–Gross relation

$$2xF_1(x, Q^2) = F_2(x, Q^2). \tag{17.9}$$

At this level it is less important to memorize the full hadronic tensor decomposition than to understand what these objects mean: they are experimentally measurable summaries of proton structure.

Remark 17.1: How formal should DIS be in this module?

For Module 4 the goal is not a full tensor-level derivation of the DIS cross section. The goal is to understand why the measured structure functions encode partonic proton structure and why their approximate scaling and subsequent scaling violation became decisive evidence for QCD.

17.5 Bjorken scaling and scaling violation

One of the most important early discoveries in DIS was that structure functions depend approximately on x alone over broad kinematic ranges. In the simplest parton model this is Bjorken scaling:

$$F_i(x, Q^2) \to F_i(x). \tag{17.10}$$

Exact scaling is not observed. Instead, the structure functions show a logarithmic Q^2 dependence. This is called scaling violation, and in modern QCD it is precisely what one expects once gluon radiation and the running of α_s are taken into account. The important conceptual point is that approximate Bjorken scaling and its violation should be read together, not in opposition: approximate scaling indicates that the proton contains point-like constituents, while the systematic Q^2 dependence shows that these constituents are embedded in an interacting non-Abelian gauge theory rather than in a strictly free parton picture.

17.6 Neutral-current and charged-current DIS

Conceptually, DIS can proceed through neutral-current exchange, dominated by γ^* and at sufficiently high scale by Z exchange, or through charged-current exchange mediated by $W^\pm$ bosons. The neutral-current process is the cleanest starting point for introducing the parton picture because the coupling to electric charge is transparent. Charged-current DIS is nevertheless important because it probes different flavor combinations and reveals the electroweak structure of the Standard Model inside the DIS framework.

17.7 HERA as an illustrative example

The HERA electron–proton collider made the modern DIS picture especially vivid. It extended deep inelastic scattering measurements into a regime of very high Q^2 and very small x, thereby providing direct experimental access to both short-distance proton structure and the parton dynamics relevant at low momentum fraction. It also made clear at the event level how the scattered electron and the hadronic final state play complementary roles: the electron kinematics are used to reconstruct variables such as Q^2 and x, while the hadronic system reveals the struck-parton dynamics and the resulting jet activity.

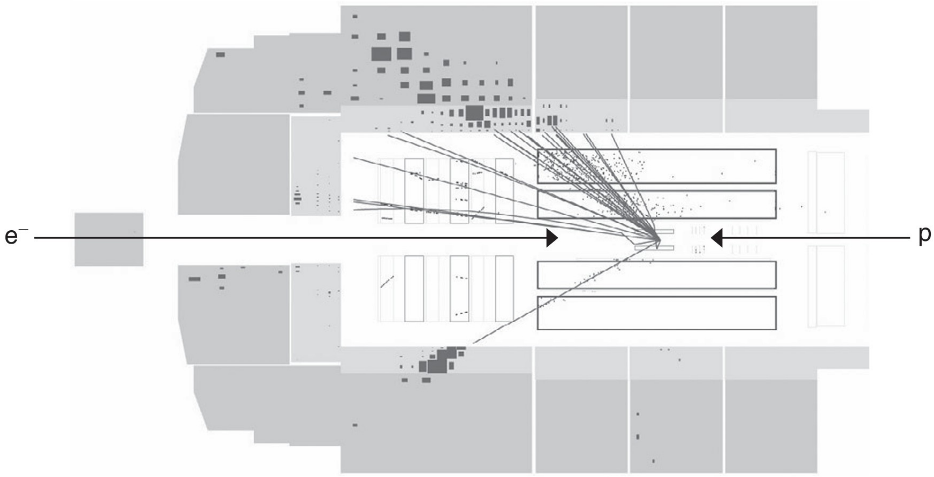


Figure 14: A high-Q^2 electron–proton collision at HERA, illustrating how DIS reveals a hard-scattering electron together with a hadronic jet system.

Guided checks

1. Why is $Q^2 = -q^2$ defined to be positive in DIS?
2. In what sense does Bjorken x become a momentum fraction in the parton picture?
3. Why was scaling violation an argument *for* QCD rather than against the parton model?

Take-home message
Deep inelastic scattering is the cleanest bridge from the formal QCD language of quarks and gluons to the measured internal structure of the proton.

18 Parton distribution functions and scaling violation

18.1 What a PDF represents

Parton distribution functions quantify how the proton momentum is distributed among its quarks, antiquarks, and gluons. At introductory level one writes, for example, $u_p(x, Q^2)$ for the up-quark PDF of the proton and interprets

$$u_p(x, Q^2)\, dx \tag{18.1}$$

as the number density of up quarks carrying momentum fraction between x and $x + dx$ at the probing scale Q^2.

Definition 18.1: Parton distribution function

A parton distribution function (PDF) describes the momentum-fraction distribution of quarks, antiquarks, or gluons inside a fast hadron at a specified factorization scale. In practice PDFs are nonperturbative inputs extracted from data.

18.2 Valence quarks, sea quarks, and gluons

At qualitative level the proton contains three intertwined components:

- valence quarks, which determine the net quantum numbers,
- sea quarks and antiquarks, generated dynamically,
- gluons, which carry a substantial fraction of the proton momentum.

We should emphasize that the proton is a complicated many-body QCD system, not a rigid "three-particle" object. The schematic figure below is useful for emphasizing that the proton should be viewed as a dynamical many-body system whose quark and gluon content can be sampled differently from event to event.

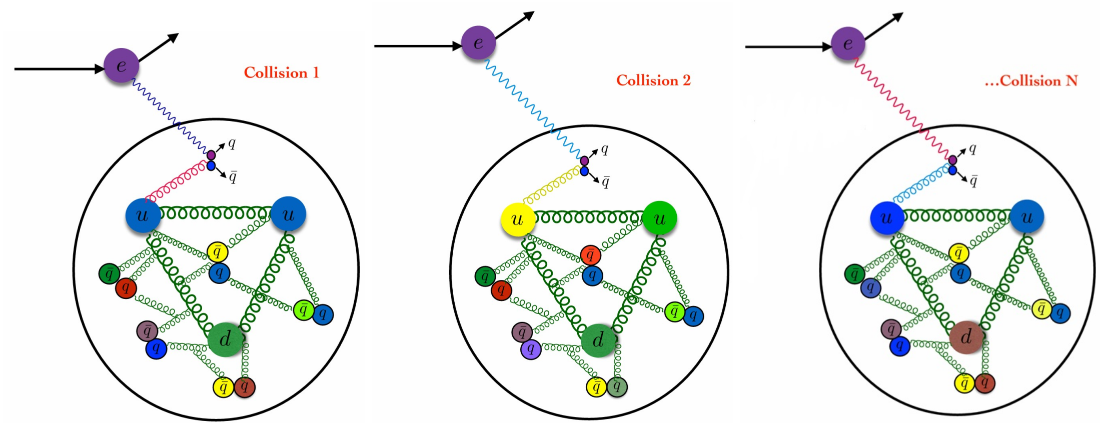


Figure 15: Schematic illustration of the proton as a dynamical many-body QCD system. Different hard collisions can probe different quark and gluon configurations inside the proton, highlighting the roles of valence quarks, sea quarks, and gluons in its partonic structure. The figure is intended as a conceptual visualization rather than a literal spatial image of the proton interior.

18.3 Why gluons are inferred indirectly

In the simplest neutral-current DIS picture the exchanged photon couples directly to electric charge, so the most direct sensitivity is to quark and antiquark distributions. Gluons enter more indirectly:

- through the Q^2 evolution of quark distributions,
- through higher-order final states,
- through hard processes in hadron collisions where gluon-initiated channels are dominant.

This is an important pedagogical point: gluons are very real, but they are often inferred through the pattern of scaling violation and hard-process systematics rather than by a literal tree-level γg coupling in the simplest DIS picture.

18.4 Scale dependence of PDFs

PDFs are not fixed functions of x alone. They depend on the probing scale:

$$f_i(x) \to f_i(x, Q^2). \tag{18.2}$$

As the scale increases, additional radiation is resolved and the momentum distribution among quarks, antiquarks, and gluons changes. Two basic sum-rule ideas are pedagogically useful. For the proton,

$$\int_0^1 \mathrm{d}x\, [u(x) - \bar{u}(x)] = 2, \qquad \int_0^1 \mathrm{d}x\, [d(x) - \bar{d}(x)] = 1, \tag{18.3}$$

and the total proton momentum is shared among all partons,

$$\sum_i \int_0^1 \mathrm{d}x\, x f_i(x, Q^2) = 1. \tag{18.4}$$

18.5 Why collider predictions require PDFs

A hadron collider does not collide elementary indivisible protons in the hard subprocess. It collides partons sampled from proton wavefunctions. Consequently, any proton–proton hard-process prediction requires PDFs as nonperturbative inputs. A schematic factorized expression is

$$\sigma(pp \to X) = \sum_{a,b} \int \mathrm{d}x_1\, \mathrm{d}x_2\, f_{a/p}(x_1, \mu_F^2) f_{b/p}(x_2, \mu_F^2)\, \hat{\sigma}_{(ab\to X)}; (x_1, x_2, \mu_F, \mu_R). \tag{18.5}$$

Here

- f_a and f_b are PDFs,
- x_1 and x_2 are the momentum fractions of the incoming partons,
- $\hat{\sigma}$ is the short-distance partonic cross section,
- μ_F and μ_R are the factorization and renormalization scales.

This formula is one of the central conceptual bridges between the hadronic world and perturbative QCD.

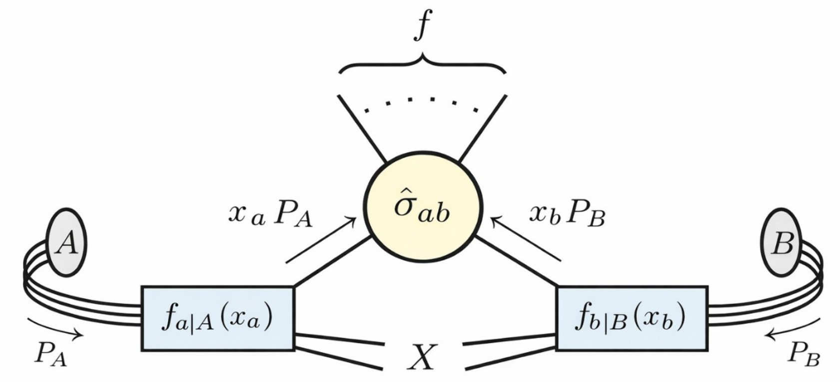


Figure 16: Schematic factorization picture for a hadron–hadron hard process. The incoming hadrons A and B are described through parton distribution functions $f_{a|A}(x_a)$ and $f_{b|B}(x_b)$, while the short-distance interaction is encoded in the partonic cross section $\hat{\sigma}_{ab}$. The observed hadronic final state X is obtained after the hard subprocess is embedded in the full QCD event structure.

The schematic factorization picture (see Fig. 16) is worth visualizing because it makes clear that a hadron–hadron prediction is not built from one indivisible proton-level interaction. Instead, each incoming proton supplies a parton carrying a momentum fraction x_1 or x_2, described by the corresponding parton distribution function. These long-distance, nonperturbative proton-structure effects are then combined with a short-distance hard subprocess, represented by the partonic cross section $\hat{\sigma}$. In this way, the observable hadronic cross section is constructed from a convolution of universal proton-structure information with perturbatively calculable parton-level dynamics.

The figure should also be read as a statement about scales. The PDFs encode long-distance structure associated with the proton wavefunction, while the hard interaction probes a much shorter distance and can be treated within perturbative QCD provided that the characteristic scale of the process is sufficiently large. The hadronic final state X then reminds us that the experimentally observed event is not a bare partonic configuration, but the result of QCD radiation, color flow, and hadronization acting on the underlying hard scattering.

At the level of this module, the most important lesson is conceptual rather than formal: factorization is the bridge that allows the partonic language of QCD to generate actual predictions for proton–proton collisions. Without PDFs, the hard subprocess alone would be incomplete; without the perturbative hard subprocess, the proton structure alone would not determine the measured high-energy scattering rate.

18.6 A brief orientation remark on evolution equations

The quantitative scale dependence of PDFs is governed by evolution equations, usually called the DGLAP equations. A full derivation is beyond the scope of this module. At the level of physical interpretation, however, the message is simple:

- partons radiate,
- the partonic momentum distribution therefore changes with scale,
- QCD predicts the pattern of that change.

Guided checks

1. Why are PDFs nonperturbative inputs even when the hard subprocess is perturbative?
2. Why is the gluon distribution often inferred through scaling violation rather than direct tree-level DIS coupling?
3. In Eq. (18.5), which pieces are long-distance and which pieces are short-distance?

Take-home message
PDFs are the language that translates the partonic structure of the proton into actual predictions for hadron collider observables.

19 Hadronization and jets

19.1 Why partons are not observed directly

In a hard scattering process the short-distance interaction may produce energetic quarks or gluons, but these partons do not propagate to the detector as free asymptotic states. As the partons separate, the strong coupling increases and the system evolves into color-singlet hadrons. This nonperturbative reorganization is called hadronization.

19.2 From parton showers to hadrons

A useful two-stage picture is the following.

1. At intermediate scales the energetic outgoing parton undergoes perturbative QCD radiation, producing a parton shower.

2. At low scales the shower enters the nonperturbative regime and the colored partons reorganize into hadrons.

This language is practical rather than exact, but it is pedagogically valuable because it explains how a single hard quark or gluon can lead to many hadrons in the detector.

The figure below provides a qualitative picture of how hadronization can emerge from confinement-driven string breaking. As energetic colored partons move apart, the color field between them is concentrated into tube-like regions. Once enough energy is stored, new $q\bar{q}$ pairs can be created, breaking the string into shorter pieces. Repetition of this process leads to the formation of multiple color-singlet hadrons, which experimentally appear as jets.

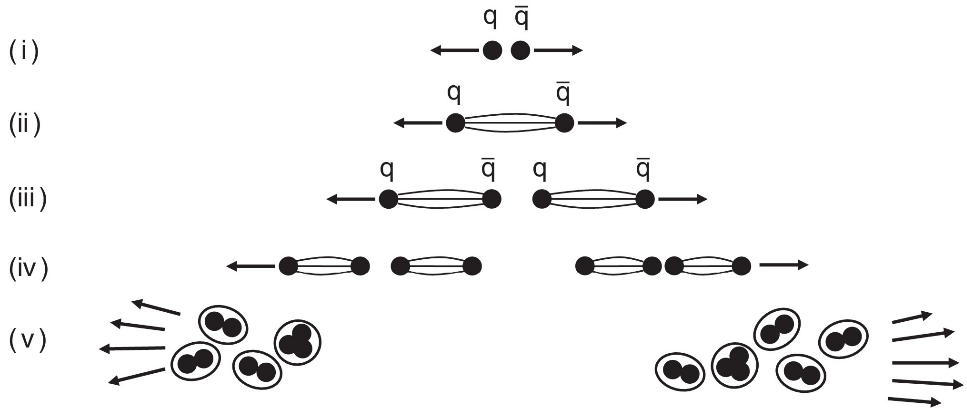


Figure 17: Qualitative picture of the steps in the hadronization process. A separating $q\bar{q}$ system forms a color tube; repeated $q\bar{q}$ creation then breaks the string into shorter segments until colorless hadrons are formed.

Remark 19.1: Parton shower versus hadronization

A parton shower belongs mainly to perturbative QCD, where successive emissions can be organized and modeled. Hadronization belongs to the nonperturbative regime. They are connected stages of one event, but they are not the same physics.

19.3 Jets as the experimental imprint of partons

Because the hadrons produced in the shower and hadronization process remain approximately collimated around the original parton direction, the detector sees a localized spray of tracks and calorimeter energy called a jet. A jet is therefore not the direct detection of a free quark or free gluon; it is the experimentally visible remnant of a hard parton.

19.4 Quark jets versus gluon jets qualitatively

At qualitative level, gluon jets tend to be broader and more particle-rich than quark jets because gluons radiate more strongly. For the purposes of this module it is enough to understand this as a physical trend rather than to develop a quantitative jet-substructure framework.

19.5 Why jets matter pedagogically

Jets are where QCD becomes experimentally tangible. They are not accidental clutter surrounding the "real" physics. They are the direct detector-level imprint of short-distance quark and gluon production.

Example 19.1: Jets connect confinement to observables

Confinement says that isolated partons are not observed. Hard scattering says that quarks and gluons are the relevant short-distance degrees of freedom. Jets are precisely the bridge between these two statements: they are what short-distance partons become when confinement and hadronization do their work.

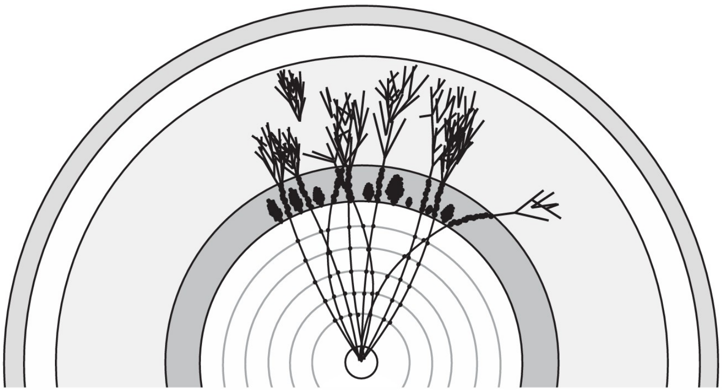


Figure 18: Illustrative detector-level appearance of a jet: individual particles need not be resolved for the collimated hadronic spray to be identified as one reconstructed object.

Guided checks

1. Why is a jet not evidence against confinement?
2. What is the conceptual difference between a parton shower and hadronization?
3. Why are jets among the most visible experimental signatures of QCD?

Take-home message

Jets are the experimental language in which short-distance quarks and gluons become visible without ever appearing as free asymptotic colored particles.

20 QCD in electron–positron annihilation

Electron–positron annihilation provides one of the cleanest environments in which to see QCD at work. The initial state is purely leptonic, so there are no proton PDFs, no beam remnants, and no underlying hadronic structure obscuring the relation between the short-distance hard process and the observed final-state topology. For this reason, e^+e^- collisions give a particularly transparent bridge between quarks, gluons, and jets.

20.1 The basic two-jet process

At leading order the hard process produces a quark–antiquark pair,

$$e^+e^- \to q\bar{q}. \tag{20.1}$$

Because quarks are not observed as free asymptotic particles, the produced q and $\bar{q}$ undergo showering and hadronization. Experimentally, this appears as a two-jet final state,

$$e^+e^- \to q\bar{q} \to 2 \text{ jets}. \tag{20.2}$$

This is shown schematically in Fig. 19a. The two-jet topology is the simplest visible manifestation of the underlying quark–antiquark hard process.

20.2 Gluon radiation and three-jet events

At next order in QCD, one of the outgoing quarks can radiate a gluon,

$$e^+e^- \to q\bar{q}g, \tag{20.3}$$

which after hadronization becomes a three-jet event,

$$e^+e^- \to q\bar{q}g \to 3 \text{ jets}. \tag{20.4}$$

This topology is shown in Fig. 19b. Historically, the observation of such three-jet events was a decisive piece of evidence for the gluon as a real dynamical degree of freedom rather than a merely formal gauge field.

It is pedagogically useful to view the two- and three-jet cases together. The same underlying annihilation process can produce progressively richer final states once QCD radiation is included. In this sense, the passage from panel (a) to panel (b) in Fig. 19 is the visible collider manifestation of gluon emission from the final-state quark or antiquark.

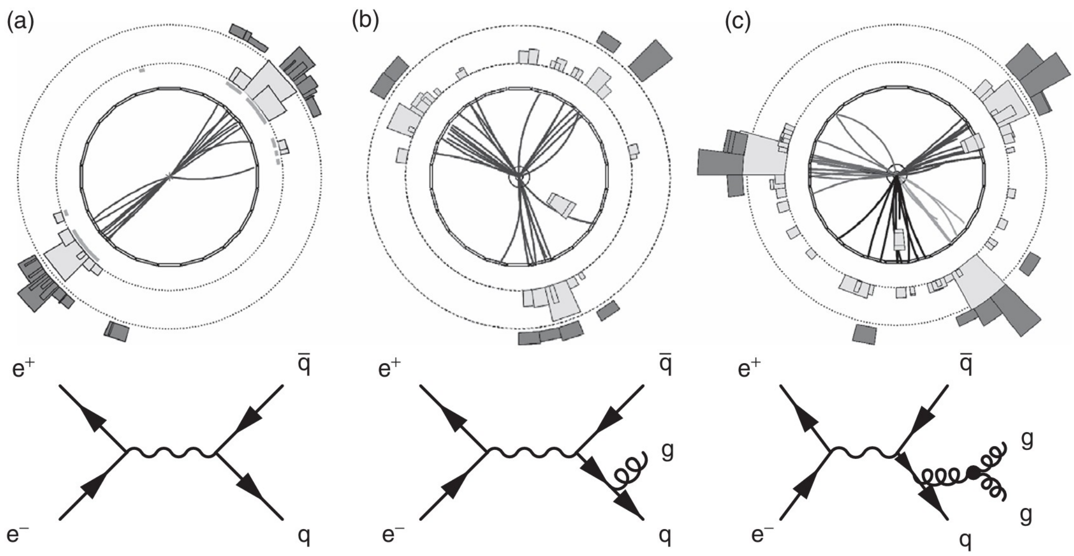


Figure 19: Jet production in e^+e^- annihilation. The example OPAL events correspond to (a) $e^+e^- \to q\bar{q} \to$ two jets, (b) $e^+e^- \to q\bar{q}g \to$ three jets, and (c) $e^+e^- \to q\bar{q}gg \to$ four jets. Also shown are possible Feynman diagrams corresponding to the observed topologies. The figure is especially useful pedagogically because it makes visible how increasingly rich jet final states emerge once QCD radiation is included.

The full figure also includes a four-jet example, shown in Fig. 19c. While this goes beyond the minimal discussion needed here, it is useful because it shows that e^+e^- annihilation does not stop at the two-jet or three-jet level: once additional QCD radiation is present, even richer final-state topologies arise. In particular, four-jet production gives access to features of the non-Abelian gauge structure of QCD, including diagrams involving the triple-gluon vertex.

20.3 Why e^+e^- is a clean QCD environment

The initial state in e^+e^- annihilation is leptonic. There are no proton PDFs and no hadronic beam remnants. This makes the relation between the short-distance hard process and the final jet topology much cleaner than in a hadron collider environment. For teaching purposes it is therefore often the best first place to see how quarks, gluons, and jets are connected.

20.4 A note on the R ratio

A classic observable is

$$R(s) \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}. \tag{20.5}$$

At simple parton-model level,

$$R(s) \approx N_c \sum_{q,2m_q \lesssim \sqrt{s}} e_q^2, \tag{20.6}$$

when the center-of-mass energy is above the relevant flavor thresholds. This makes R a beautifully compact example of how an experimentally measurable quantity reveals a deep structural fact of QCD: quarks come in three colors.

Example 20.1: Why the factor N_c matters

If quarks came in only one color, the hadronic production rate in e^+e^- annihilation would be too small by a factor of three relative to what is observed. The R ratio is therefore one of the classic arguments that the color multiplicity is physically real even though free color states are never seen.

21 QCD at hadron colliders

21.1 Proton–proton collisions as parton–parton collisions

At a hadron collider such as the LHC, the incoming beams are protons but the hard interaction occurs between partons drawn from those protons. If the proton–proton center-of-mass energy is $\sqrt{s}$, then the hard subprocess has squared center-of-mass energy

$$\hat{s} = x_1 x_2 s, \tag{21.1}$$

where x_1 and x_2 are the momentum fractions of the colliding partons.

21.2 Why hadron-collider kinematics is richer

In DIS there is one hadron in the initial state; in pp collisions there are two, and both provide partonic momentum fractions that are not known event by event. Even a simple heavy system of invariant mass M already satisfies

$$x_1 x_2 = \frac{M^2}{s}, \tag{21.2}$$

while its rapidity is related to the imbalance of the two incoming partons through

$$y = \frac{1}{2} \ln\left(\frac{x_1}{x_2}\right). \tag{21.3}$$

This is one reason hadron-collider phenomenology is richer and more involved than either e^+e^- annihilation or the simplest DIS picture.

21.3 Transverse momentum

Because the incoming beams define a special longitudinal direction, transverse observables are central. The transverse momentum is

$$p_T = \sqrt{p_x^2 + p_y^2}. \tag{21.4}$$

Large p_T is typically a signal of a hard short-distance scattering.

21.4 Rapidity and pseudorapidity

The natural angular variable in collider kinematics is often rapidity,

$$y = \frac{1}{2} \ln\left(\frac{E + p_z}{E - p_z}\right), \tag{21.5}$$

because differences in rapidity are invariant under boosts along the beam axis. When the mass of the observed object can be neglected relative to its energy, one often uses the pseudorapidity

$$\eta \equiv -\ln\left(\tan\frac{\theta}{2}\right). \tag{21.6}$$

Definition 21.1: Factorization in hadron collisions

Factorization is the statement that, for sufficiently hard inclusive processes, the hadronic cross section can be separated into long-distance proton-structure information encoded in PDFs and a short-distance partonic cross section calculable in perturbative QCD.

21.5 Factorized picture of a hadron-collider event

A realistic hadron-collider event combines several layers of physics:

- proton structure encoded in PDFs,
- a perturbative hard subprocess,
- initial-state and final-state QCD radiation,
- beam remnants and underlying activity,
- hadronization into observable jets and hadrons.

This is why QCD is present from the initial state onward, even when the “headline” hard interaction is electroweak.

The figure below is intentionally broader than the factorized hard subprocess alone. It shows the layered structure of a realistic hadron-collider event: parton distributions set the incoming proton structure, a short-distance hard scattering occurs at high scale, QCD radiation builds a parton shower, and the final colored partons are reorganized into hadrons before unstable particles decay. In this way, the observed event emerges from several physically distinct stages rather than from a single isolated interaction.

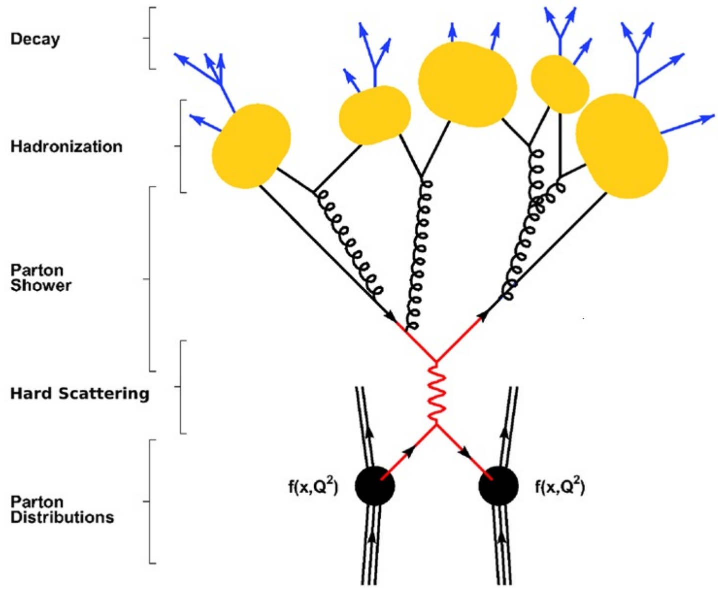


Figure 20: Schematic view of a hadron-collider event, showing the successive layers from parton distributions and hard scattering to parton showering, hadronization, and particle decays. The figure is useful for emphasizing that a realistic proton–proton event contains much more than the short-distance subprocess alone.

21.6 Jets as dominant final states

Because QCD is both strong and ubiquitous in proton collisions, jets are among the most common visible final states at the LHC. Even when the hard process is not itself "pure QCD," initial-state radiation, final-state radiation, and hadronic recoil often shape the event topology decisively.

Example 21.1: Why an electroweak process is still a QCD event environment

In Drell–Yan production, the hard subprocess is electroweak, $q\bar{q} \to \gamma^*/Z \to \ell^+\ell^-$. Yet the observed event still depends on PDFs, initial-state radiation, and accompanying jet activity. This is why one must learn to read QCD not only as signal physics but also as event environment.

Guided checks

1. Why does $\hat{s} = x_1 x_2 s$ immediately tell you that partonic collisions occur at lower energy than the proton beam energy?
2. Why are rapidity and pseudorapidity more natural than a simple polar angle in collider physics?
3. In what sense is factorization the conceptual bridge from proton structure to perturbative predictions?

Take-home message

At a hadron collider, QCD is not an optional complication added on top of the real physics. It is part of the real physics from the initial state onward.

22 Representative major QCD-sensitive processes at the LHC

22.1 Inclusive jet production

Inclusive jet production is the most direct large-rate manifestation of hard QCD at the LHC. The observed jet cross section receives contributions from quark–quark, quark–gluon, and gluon–gluon hard scattering, followed by radiation and hadronization. Pedagogically it is valuable because it is both simple in concept and central in practice: if students want to see QCD "in bulk," inclusive jets are one of the clearest places to look.

The figures below connect the parton-level and experimental faces of inclusive jet production. The first summarizes representative lowest-order QCD subprocesses contributing to jet production in proton–proton collisions, while the second shows the corresponding inclusive jet spectrum measured at the LHC. Together they illustrate how hard quark–quark, quark–gluon, and gluon–gluon scattering becomes a directly measurable collider observable.

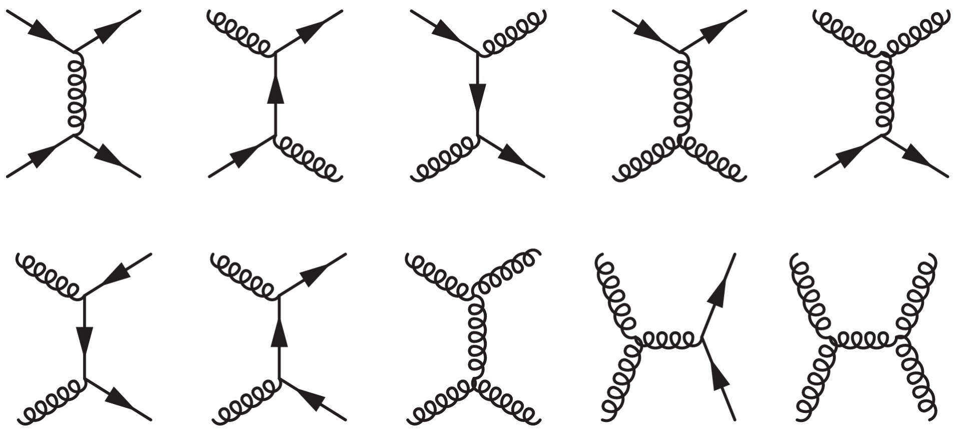


Figure 21: Representative lowest-order parton-level subprocesses contributing to two-jet production in proton–proton collisions.

A useful next step is to see that this parton-level picture is not merely schematic: inclusive jet production is measured with high precision at the LHC and provides one of the clearest large-rate tests of perturbative QCD.

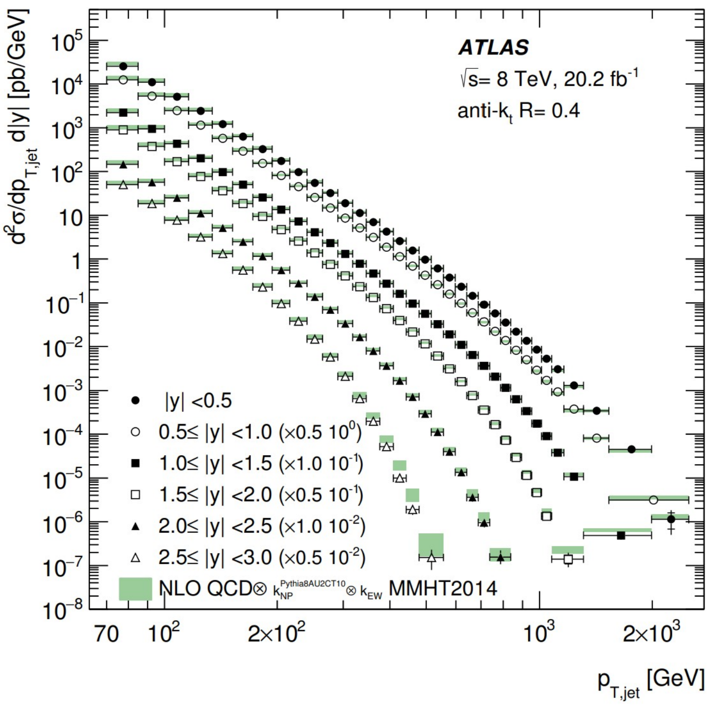


Figure 22: Inclusive jet production cross section measured by ATLAS at the LHC, compared with QCD predictions [JHEP 09 (2017) 020].

22.2 Dijet production

Dijet events are the cleanest hadron-collider analogue of hard two-body parton scattering. Their invariant mass, angular distributions, and rapidity structure reveal the underlying parton kinematics, while the overall rate probes the interplay of PDFs and short-distance QCD matrix elements.

Dijet production is especially useful because it gives more direct access to two-body parton kinematics. Observables such as the dijet invariant mass, rapidity separation, and angular distributions provide a clean bridge between perturbative parton scattering and experimentally reconstructed jet events.

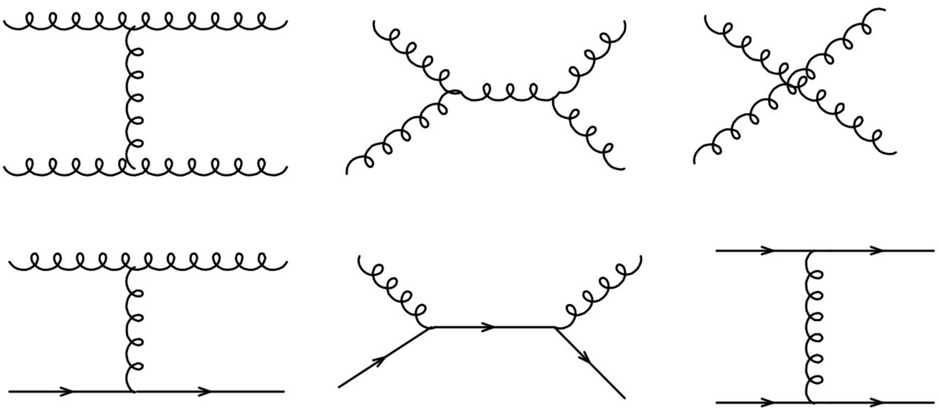


Figure 23: Representative diagrams for the production of jet pairs in hadronic collisions.

22.3 Multijet production

Once gluon radiation is included, multijet final states arise naturally. They are not exotic exceptions but a generic consequence of QCD. Their importance is pedagogical as well as practical: they make visible the fact that hard scattering and QCD radiation are not separate worlds but successive layers of the same event.

22.4 Heavy-quark production

QCD also drives the production of heavy quarks, especially charm, bottom, and top. Heavy-quark production is interesting because it combines hard perturbative scales with flavor-specific final states, displaced decays, and the possibility of tagging heavy-flavor jets.

22.5 Top-pair production as a gluon-rich process

At the LHC, top-pair production receives a large contribution from gluon–gluon initial states ($\sim 80-90\%$),

$$gg \to t\bar{t}. \tag{22.1}$$

This makes top-pair production a vivid example of the central role of gluons in modern proton–proton collider physics.

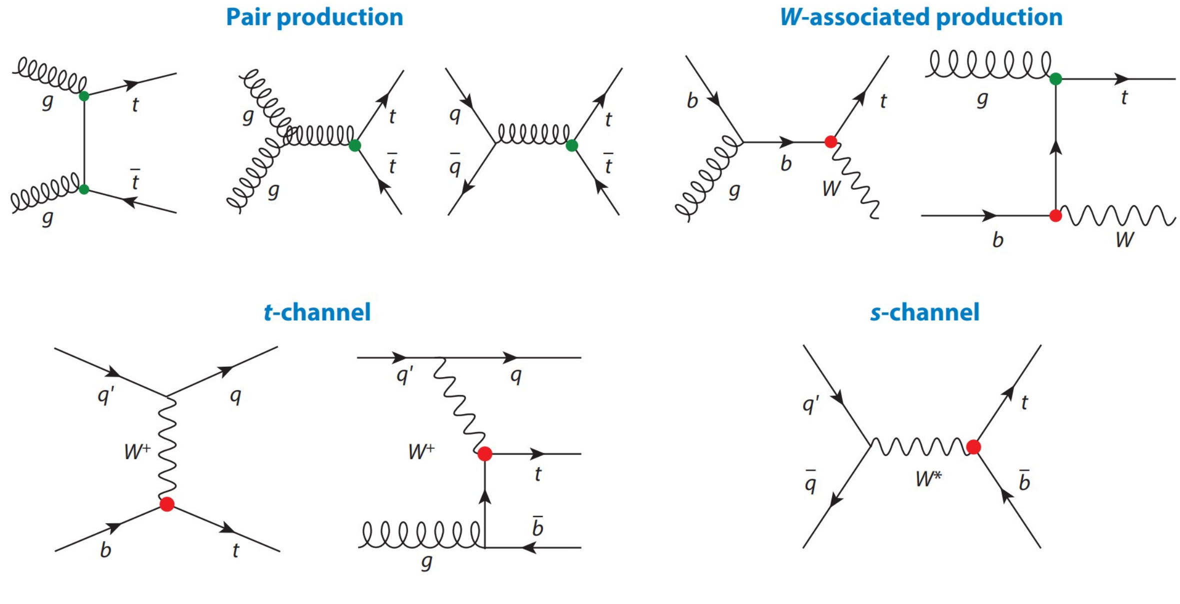


Figure 24: Representative diagrams for strong production of top quark–antiquark pairs, electroweak single top quark production in the t- and s-channels, and W boson–associated production. The top quark couplings to gluons are indicated by green circles; couplings to W bosons are indicated by red circles.

Top production provides a particularly instructive example of modern collider QCD. In proton–proton collisions, top–antitop production receives a large gluon-initiated contribution, so it makes the importance of the gluon content of the proton especially visible. At the same time, the broader set of top-production channels reminds us that collider processes are rarely "purely QCD" or "purely electroweak": rather, they reflect the interplay of proton structure, hard scattering, and the couplings of the produced heavy particle.

Fig. 24 is therefore useful not only as an illustration of $gg \to t\bar{t}$, but also as a compact map of the main production mechanisms through which top quarks appear at hadron colliders.

22.6 Drell–Yan plus QCD radiation

The Drell–Yan process,

$$q\bar{q} \to \gamma^*/Z \to \ell^+\ell^-, \tag{22.2}$$

is electroweak at the hard-interaction level, but realistic events carry PDFs, recoil, initial-state radiation, and accompanying jet activity. It is therefore an excellent example of a process whose "signal" is not QCD but whose detailed event structure is still shaped by QCD.

The figure below is useful because it shows the Drell–Yan process in the language of the parton model and factorization. Although the observed initial state is hadronic, the short-distance hard interaction is the annihilation of a quark from one projectile with an antiquark from the other into a virtual electroweak boson, followed by decay into a lepton pair. In this way, Drell–Yan production provides a particularly clean example of how proton structure, encoded in PDFs, is combined with a perturbatively calculable partonic subprocess to produce an experimentally measurable hadronic-collider observable:

$$\sigma^{\mathrm{DY}} \sim \sigma\left(\bar{q}q \to e^+e^-\right) \otimes q(x_1) \otimes \bar{q}(x_2). \tag{22.3}$$

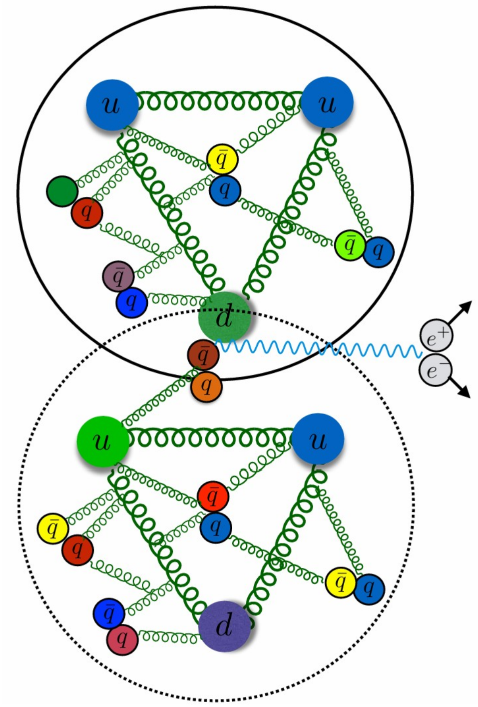


Figure 25: Schematic factorized view of the Drell–Yan process in hadron–hadron collisions. A quark from one incoming hadron and an antiquark from the other annihilate into a virtual electroweak boson, which then produces an e^+e^- pair. The figure also emphasizes the parton-model logic that the hadronic cross section is built from the hard subprocess $\bar{q}q \to e^+e^-$ folded with the relevant parton distributions $q(x_1)$ and $\bar{q}(x_2)$.

Although the hard subprocess in Drell–Yan production is electroweak, the event is embedded in a hadronic environment from the start. The annihilating quark and antiquark are extracted from extended hadronic projectiles, and the remnants of those projectiles continue into the final state as colored QCD matter that later hadronizes. In this sense, Drell–Yan production is a particularly clean electroweak signal inside a genuinely QCD-controlled collider event: the short-distance vertex is simple, but the full event structure still reflects proton structure, initial-state radiation, recoil, and hadronic activity.

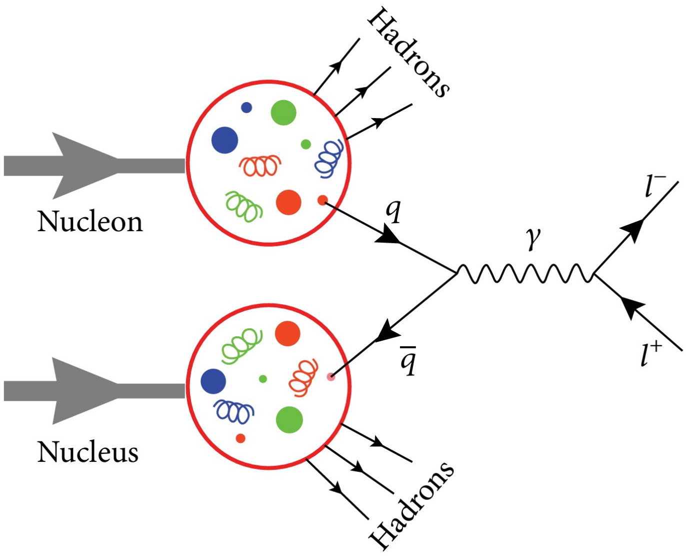


Figure 26: Schematic illustration of the Drell–Yan process for a nucleon–nucleus collision. A quark and antiquark annihilate into a virtual photon that decays into a lepton pair, while the remaining colored constituents continue as hadronic remnants. The figure highlights the contrast between the clean electroweak hard vertex and the surrounding QCD event environment.

22.7 Higgs production via gluon fusion

A flagship Standard Model process at the LHC is Higgs production via gluon fusion,

$$gg \to H, \tag{22.4}$$

mediated at leading order predominantly through a top-quark loop in the Standard Model. This is particularly striking pedagogically because it shows that gluons are not merely responsible for strong-interaction backgrounds; they also dominate the rate of one of the most important Standard Model processes.

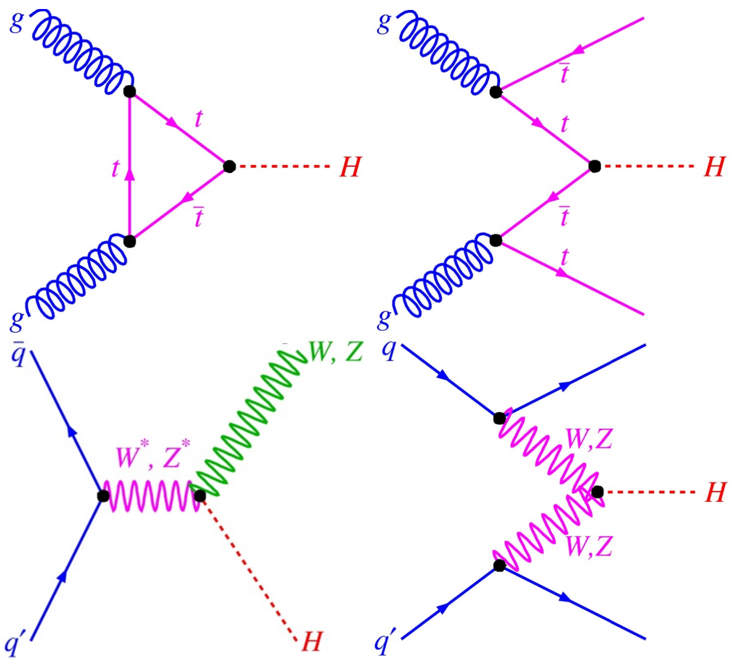


Figure 27: Sketch of the main Higgs production mechanisms at the LHC: gluon fusion, associated production with a top–antitop pair, Higgs-strahlung with electroweak gauge bosons, and vector-boson fusion. The figure is pedagogically useful because it shows that, although gluon fusion is the dominant production mode, several distinct channels contribute to Higgs phenomenology at hadron colliders.

Although this subsection emphasizes gluon fusion, the figure above is useful because it places $gg \to H$ within the wider set of leading Higgs-production mechanisms at the LHC. In the Standard Model, gluon fusion is the dominant production mode and proceeds through a heavy-quark loop, predominantly the top quark. This is one of the clearest examples of how the gluon content of the proton controls not only strong-interaction backgrounds, but also the rate of a central electroweak process.

At the same time, Higgs production is not exhausted by gluon fusion alone. Associated production with top quarks, Higgs-strahlung with electroweak gauge bosons, and vector-boson-fusion channels all provide complementary experimental signatures and are important for different measurement strategies. Read in this way, the figure is not just a catalogue of diagrams: it shows how proton structure, QCD initial states, heavy-particle loops, and electroweak couplings all enter the Higgs sector at hadron colliders.

22.8 QCD as signal, background, and event environment

At the LHC, QCD plays three distinct but overlapping roles:

1. **signal**, when the hard process itself is a QCD scattering,
2. **background**, when QCD events mimic or obscure rarer signals,
3. **environment**, when QCD radiation and hadronization shape the observed event around another hard process.

This tripartite role is one of the main reasons QCD is unavoidable for any serious engagement with collider phenomenology or data analysis.

Take-home message
Modern collider physics is impossible to understand without QCD. QCD determines proton structure, shapes event radiation, produces the dominant visible final states, and often controls the largest cross sections in the experiment.

23 What QCD teaches us about the Standard Model

23.1 QCD is more than "the strong-force chapter"

By this point, QCD should no longer look like one more sector in a particle table. It is the Standard Model sector in which abstract gauge symmetry, hadronic matter, and collider observables become inseparable. It ties together

- the non-Abelian gauge principle,
- the existence of color-singlet matter,
- the internal structure of hadrons,
- the partonic picture of the proton,
- the ubiquity of jets in collider experiments.

23.2 Visible matter and the dominance of QCD

Most of the mass of ordinary visible matter is tied far more to QCD dynamics than to a naive sum of current quark masses. In this sense the hadronic world around us is not peripheral to the Standard Model; it is one of its most tangible physical realizations.

23.3 Gauge symmetry becoming experimentally visible

One of the deepest lessons of Module 4 is that the abstract color gauge symmetry

$$\mathrm{SU}(3)_c \tag{23.1}$$

is not a merely formal shell. It becomes experimentally visible through the running of α_s, asymptotic freedom, confinement, hadron structure, jet production, and the event environment of modern colliders.

23.4 QCD as the bridge from Lagrangian to collider reality

In the overall course logic,

$$\text{formal language} \to \text{fields and symmetries} \to \text{interactions} \to \text{observables} \to \text{data},$$

QCD occupies a uniquely rich place. It begins as a compact Yang–Mills Lagrangian and unfolds into a vast range of experimentally visible phenomena.

24 Bridge to later modules

24.1 From QCD to electroweak symmetry breaking: Module 5

Module 4 has shown how one sector of the Standard Model gauge symmetry becomes a full physical theory of strong interactions. The next module asks a different question: how the electroweak sector can generate massive gauge bosons and fermions without destroying the internal consistency of the theory. That is the role of spontaneous symmetry breaking and the Higgs mechanism.

24.2 From QCD structure to amplitudes and observables: Module 6

Module 4 has already used process-level language in DIS, e^+e^- annihilation, and hadron-collider examples, but only in service of the QCD story. Module 6 will treat more systematically the broader machinery by which one goes from a Lagrangian to amplitudes, decays, and cross sections.

24.3 From collider signatures to data analysis: Module 8

Jets, p_T distributions, rapidity spectra, inclusive rates, and detector-level event structure are not only lecture concepts. They are natural entry points for the computational and analysis work of the laboratory and project components.

25 Final summary and conceptual map

25.1 The logical chain of Module 4

The whole logic of Module 4 may now be summarized as

$$\mathrm{SU}(3)_c \longrightarrow \text{quark and gluon fields} \longrightarrow \mathcal{L}_{\mathrm{QCD}} \longrightarrow \alpha_s(Q^2) \longrightarrow \text{asymptotic freedom} \longrightarrow \text{confinement} \longrightarrow \text{hadrons} \longrightarrow$$

25.2 QCD is not just a technical sector

QCD is one of the clearest examples in modern physics of how a compact formal theory generates a rich and experimentally visible world. It is simultaneously a non-Abelian gauge theory of color, the theory of hadrons, and the theory of much of the event environment at modern colliders.

25.3 What the student should now be able to see

After completing Module 4, the student should be able to see that:

- the strong interaction is described by a specific Yang–Mills theory based on $\mathrm{SU}(3)_c$,
- the running of the strong coupling explains why perturbation theory works at short distance but fails at long distance,
- confinement reorganizes the physical spectrum into color-singlet hadrons,

- deep inelastic scattering reveals the partonic structure of the proton,
- PDFs, hadronization, and jets are central parts of QCD rather than optional complications,
- collider physics at the LHC is impossible to understand without QCD.

Take-home message
QCD is the point in the Standard Model where symmetry, dynamics, hadronic matter, and collider phenomenology become inseparable. It is therefore both a formal gauge theory and one of the most experimentally tangible sectors of modern particle physics.

A Conventions summary

- Metric: $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.
- Natural units: $\hbar = c = 1$.
- QCD coupling: $\alpha_s = g_s^2/(4\pi)$.
- Color indices: $i, j, k = 1, 2, 3$.
- Adjoint indices: $a, b, c = 1, \ldots, 8$.
- Generator normalization: $\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$.
- Lie algebra: $[T^a, T^b] = \mathrm{i} f^{abc} T^c$.

B SU(3) and color summary

B.1 Basic representation facts

$$\text{quarks:} \quad \mathbf{3}, \tag{B.1}$$
$$\text{gluons:} \quad \mathbf{8}\ \text{(adjoint)}. \tag{B.2}$$

B.2 Useful tensor products

$$3 \otimes \bar{3} = 8 \oplus 1, \tag{B.3}$$
$$3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1. \tag{B.4}$$

B.3 Color-singlet reminders

$$\text{meson singlet:} \quad \frac{1}{\sqrt{3}}(r\bar{r} + g\bar{g} + b\bar{b}), \tag{B.5}$$
$$\text{baryon singlet:} \quad \epsilon_{ijk} q^i q^j q^k. \tag{B.6}$$

C DIS and hadron-collider kinematics summary

C.1 DIS variables

$$q = k - k', \tag{C.1}$$
$$Q^2 = -q^2, \tag{C.2}$$
$$x = \frac{Q^2}{2p \cdot q}, \tag{C.3}$$
$$y = \frac{p \cdot q}{p \cdot k}. \tag{C.4}$$

C.2 Hadron-collider variables

$$\hat{s} = x_1 x_2 s, \tag{C.5}$$
$$p_T = \sqrt{p_x^2 + p_y^2}, \tag{C.6}$$
$$y = \frac{1}{2} \ln\left(\frac{E + p_z}{E - p_z}\right), \tag{C.7}$$
$$\eta = -\ln\left(\tan\frac{\theta}{2}\right). \tag{C.8}$$

D Guided-check summary

1. Why does the gauge group SU(3) imply eight gluons rather than one or nine?
2. What is the conceptual origin of the triple-gluon and quartic-gluon vertices?
3. Why is the sign of the QCD beta function physically more important than its detailed derivation in an introductory note?
4. Why does asymptotic freedom make the parton model meaningful at high Q^2?
5. Why does confinement imply that the observed hadronic spectrum is organized into color singlets?
6. In what sense is Bjorken x both an experimental DIS variable and a parton-model momentum fraction?
7. Why are PDFs unavoidable in proton–proton collisions?
8. Why are jets the experimental imprint of hard quarks and gluons rather than direct detections of free partons?